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Peirce, Clifford, and Quantum Theory

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Beginning in 1870 Charles Sanders Peirce published a series of papers on a "logic of relations," which corresponded to a linear associative algebra. This algebra is related by a linear transformation to quaternions and thus to the C(3, 0) algebra of William Kingdon Clifford. This Clifford algebra contains the Pauli matrices and so constitutes an operator basis for the nonrelativistic quantum theory of spin one-half particles. A further unification is achieved by taking the wave functions themselves to be 2×2 matrices which are Peirce logical operators and also elements of the Clifford algebra. Thus we have discovered a direct path from the Peirce logic to quantum theory. A diagrammatic method follows from the Peirce/Clifford algebraic approach and is suitable for describing particle interactions.

KEY WORDS: Peirce logic; Clifford algebra; quantum theory.

1. INTRODUCTION

Charles Sanders Peirce (1839–1914) was one of the top American physicists of the nineteenth century (see, e.g., Cattell and Brimhall (1906); Eisele (1979); Fisch (1981); Ketner (1998); Lenzen (1975)). He was internationally well known for his work on the measurement of gravitational acceleration, the use of the wavelength of light as a measurement standard (commended by Michelson and Morley (Lenzen, 1975)), stellar photometry, and the mapping of the distribution of stars in the local galaxy. He was employed as a physicist by the U.S. Coast and Geodetic Survey from 1867 to 1891 and was a member of the National Academy of Sciences from 1877. He was personally acquainted with many of the greatest European physicists of the age, including Maxwell, whom he visited at Cambridge in April of 1875 (see Fisch, 1986, p. 125).

Peirce was also a mathematician and logician of highest standing (Fisch, 1986). He exchanged ideas with such luminaries as De Morgan, Jevons, Cayley, Sylvester, and Clifford. He held a lecturership in logic at Johns Hopkins from 1879

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to 1884. One of his students was Allan Marquand, designer of an early mechanical logic machine. Peirce himself designed an electrical computing machine at Marquand's request (Ketner, 1984). He also worked out and published a positive version of what later became known as Alan Turing's limits of computability thesis (Ketner, 1988). Peirce's father, Benjamin Peirce Jr., was a long-time professor of mathematics and astronomy at Harvard, and a world authority in the area of linear algebra (Lenzen, 1968). The younger Peirce also made significant contributions to the theory of associative algebras (Peirce, 1933).

Peirce's relationship with William Kingdon Clifford (1845–1879) is particularly noteworthy. They became acquainted when both took part in the December 1870 expedition to Sicily to observe the solar eclipse. On May 4, 1875, Peirce wrote his mother from London (Fisch, 1986, p. 126),

> Today I went to the Royal Society rooms ... and I received an invitation to attend the meetings ... I afterward went to see Clifford and had a very interesting talk with him about Logic, etc. & I am going to dine there Sunday.

Clifford complimented Peirce's physical and mathematical work in several places. Especially telling is a remark by E. L. Youmans, publisher of the *Popular Science Monthly*, in a letter home from London, October 29, 1877 (see Peirce, 1933, p. xxii; Fisch, 1986, p. 126):

> Clifford ... says he is the greatest living logician, and the second man since Aristotle who has added to the subject something material, the other being George Boole ...

As if such accomplishments were not sufficient, Peirce is widely recognized as the most influential American philosopher of both the nineteenth and twentieth centuries (Ketner, 1998, p. 40). He was the founder of the school known as Pragmatism (which he later renamed Pragmaticism). He was the main precursor of John Dewey and William James, as both acknowledged (Dewey, 1938, P. 9, and James, 1938, the dedication).

While Peirce's standing in the areas of logic and philosophy is now well established, his recognition as a mathematician and physicist remains obscure at best. Only a few have realized that Peirce's ideas in physics were profoundly innovative and not only anticipated some of the major themes of twentieth-century physics, but might be significant in developing new physics in the twenty-first century. (Christiansen, 1993; Fernandez, 1993; Finkelstein, 1988, 1994, 1996; Lenzen, 1973, 1975).

In this paper we propose a new application of Peirce's thought, based not only on his physical ideas, but also on his developments in linear algebra and logic. This is in the spirit of his overarching philosophy that knowledge is a unified whole and that themes from one discipline are applicable to others.

We will begin with a well-known mathematical background from Peirce's friend Clifford, bring in Peirce's logic-based algebra, and see how this leads quite naturally to a simple version of nonrelativistic quantum mechanics. Peirce's themes are then carried further by applying a diagrammatic method which produces a new way of looking at quantum interactions. There are many potential implications of these innovations.

2. THE CLIFFORD MATRICES

For reference, we list the standard representation of the C(3, 0) (see, Göckeler and Schücker, 1987, for the classification) Clifford algebra in 2×2 complex matrix form. This compares, for example, with that given by Snygg (1997). The identity, or unit scalar, is,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1a}$$

The vectors are

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1b}$$

These are a common expression of the Pauli matrices. The multiplication rule in this algebra is $\gamma^i \gamma^j + \gamma^j \gamma^i = 2\delta^{ij}$.

The bivectors are

$$\begin{aligned} \gamma^1\gamma^2 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\gamma^3 = -\boldsymbol{k} \\ \gamma^2\gamma^3 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\gamma^1 = -\boldsymbol{i} \\ \gamma^3\gamma^1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\gamma^2 = -\boldsymbol{j} \end{aligned} \tag{1c}$$

The $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ are quaternion bases which satisfy $\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}, \boldsymbol{ii} = \boldsymbol{jj} = \boldsymbol{kk} = -1$.

The trivector or pseudoscalar is

$$\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = iI \tag{1d}$$

The use of these matrices in the quantum theory of spin one-half particles is very familiar.

3. FROM LOGIC TO ALGEBRA

The development of Peirce's logic of relations and its affiliated linear algebra can be traced through a series of papers (Peirce, 1933, paragraphs 45, 154, 252, and 328). All of these papers are to be found in Volume III of the collected works (Peirce, 1933). Modern studies of Peirce's linear algebra include those of Lenzen (1975), Brunning (1981), and Iliff (1997).

The logic of relatives might better be called the logic of how things are related to one another. This is best explained by using an example given by Peirce (1933, paragraph 124) himself.

Start with two mutually exclusive classes of individuals, say the teachers, u_1, in a school, and the pupils, u_2, in the same school. In general, there could be more than two classes, with individuals labeled u_i. These individuals are called "absolute terms" by Peirce, which we will shorten to "absolutes." Absolutes are one type of element in the logic.

A second type of element in the logic has to do with the linear transformation of absolutes. These second elements are called "relatives" or "dual relatives" by Peirce. We choose to call them "relative operators" or just "operators." (A method in this nomenclature will become apparent when we get to quantum mechanics.) For example, an operator u_{12} (written $(u_1 : u_2)$ in Peirce's notation) is called a teacher–pupil operator. In similar fashion one has a colleague (teacher–teacher) operator, u_{11}, a pupil–teacher operator, u_{21}, and a schoolmate (pupil–pupil) operator, u_{22}. These four operators are assumed to act on the two absolutes in the following way:

$$
\begin{aligned}
&u_{11}u_1 = u_1, \quad u_{11}u_2 = 0, \quad u_{12}u_1 = 0, \quad u_{12}u_2 = u_1 \\
&u_{21}u_1 = u_2, \quad u_{21}u_2 = 0, \quad u_{22}u_1 = 0, \quad u_{22}u_2 = u_2
\end{aligned}
\tag{2}
$$

A verbal statement of $u_{12}u_2 = u_1$ would be "a teacher–pupil operator acting on a pupil produces a teacher." The relation $u_{21}u_2 = 0$ would be "a pupil–teacher operator acting on a pupil produces nothing." Of course, the verbal model has a limited correspondence to reality. The mathematical properties given in (2) (see also (3) and (11)) should be consulted for the precise action of the operators.

The general rule for the action of these operators on absolutes is

$$u_{ij}u_k = \delta_{jk}u_i \tag{3}$$

This also applies to cases with more than two classes of individuals.

The operators can also act on each other according to a multiplication rule similar to (3):

$$u_{ij}u_{kl} = \delta_{jk}u_{il} \tag{4}$$

For the case of only two classes this multiplication table is given in Table I. The operator in the left column acts from the left on the operator in the top row and produces the result in the corresponding box. The operations are associative and thus representable as matrices.

Table I determines a certain linear associative algebra. In Benjamin Peirce's landmark classification of algebras (B. Peirce, 1881) this is labeled g_4 and called a form of quaternion.

Table I. Multiplication Table for Relative Operators

	u_{11}	u_{12}	u_{21}	u_{22}
u_{11}	u_{11}	u_{12}	0	0
u_{12}	0	0	u_{11}	u_{12}
u_{21}	u_{21}	u_{22}	0	0
u_{22}	0	0	u_{21}	u_{22}

The standard multiplication table for the quaternion bases is given in Table II. A convenient 2×2 matrix representation was given in (1c). There is also a very simple matrix representation for the u_{ij}'s. This appeared in a work of Lenzen (Lenzen, 1975).

$$u_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad u_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad u_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad u_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

It is easily verified that these matrices satisfy (4). The four are obviously a basis for 2×2 matrices. In particular, the Clifford matrices and the quaternion basis can be constructed as linear combinations of the u_{ij}'s, when the coefficients are allowed to be complex. The quaternion basis transformation is

$$I = u_{11} + u_{22}, \quad \boldsymbol{i} = -iu_{12} - iu_{21}, \quad \boldsymbol{j} = u_{21} - u_{12}, \quad \boldsymbol{k} = -iu_{11} + iu_{22} \tag{6}$$

Actually this is a special case of a more general transformation given by Peirce (1976, paragraph 130). It defines the most general set of matrices which satisfy the quaternion multiplication rule.

The inverse transformation of (6) expresses the u_{ij}'s in terms of either quaternions or Clifford matrices:

$$\begin{aligned} u_{11} &= \frac{1}{2}(I + i\boldsymbol{k}) = \frac{1}{2}(I + \gamma^3) \\ u_{12} &= \frac{1}{2}(i\boldsymbol{j} - \boldsymbol{j}) = \frac{1}{2}(\gamma^1 + i\gamma^2) \\ u_{21} &= \frac{1}{2}(i\boldsymbol{i} + \boldsymbol{j}) = \frac{1}{2}(\gamma^1 - i\gamma^2) \\ u_{22} &= \frac{1}{2}(I - i\boldsymbol{k}) = \frac{1}{2}(I - \gamma^3) \end{aligned} \tag{7}$$

Table II. Multiplication Table for Quaternion Bases

	I	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
I	I	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
$\boldsymbol{i}$	$\boldsymbol{i}$	$-I$	$\boldsymbol{k}$	$-\boldsymbol{j}$
$\boldsymbol{j}$	$\boldsymbol{j}$	$-\boldsymbol{k}$	$-I$	$\boldsymbol{i}$
$\boldsymbol{k}$	$\boldsymbol{k}$	$\boldsymbol{j}$	$-\boldsymbol{i}$	$-I$

So the u_{ij} operators are not quaternion bases, but are numbers in the quaternion system which are simple linear combinations of those bases. They are also simple linear functions of Clifford vectors and are themselves also Clifford numbers.

Having gained a picture of the operators and their place in the world we now turn back to the absolutes, u_i.

We would like to be able to place the absolutes into the above algebraic scheme. This would involve expressing absolutes in terms of the relative operators and vice-versa. Actually, Peirce solved half of this problem by giving an expression for absolutes as a sum of operators,

$$u_i = \sum_j u_{ij} \tag{8}$$

This equation appears in the form $A = A : A + A : B + \cdots$ (paragraphs 220, 222, and Peirce, 193:311). For the case of two classes of individuals and the matrix representation (5), this becomes,

$$u_1 = u_{11} + u_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{9a}$$

$$u_2 = u_{21} + u_{22} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \tag{9b}$$

A sample logic of relations verbal statement of the first of these would be, "a teacher is the sum of a colleague (teacher–teacher operator) and a teacher (teacher–pupil operator)."

Although Peirce indicates by the notation $\{u_i : u_j\}$ that relative operators are *logically* related to their constituent absolutes, he specifically states (Peirce, 1933, paragraph 144) that an operator cannot *algebraically* be reduced to a combination of absolutes. Strictly speaking, this is true. But about this time Charles Hermite discovered an easy prescription to form a suitable combination of absolutes which does produce the relative operators.

Such a prescription can be constructed using the Hermitian conjugate matrices

$$u_1^* = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad u_2^* = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{10}$$

The Hermitian conjugate is a transpose of the matrix as well as a complex conjugate of the elements. Peirce, probably unaware of Hermite's work, actually defines the transpose (he calls it the "converse") but does not apply it to this problem.

The expressions for operators in terms of absolutes are then

$$u_{11} = \frac{1}{2}u_1u_1^* \qquad u_{12} = \frac{1}{2}u_1u_2^*$$

$$u_{21} = \frac{1}{2}u_1u_1^* \qquad u_{22} = \frac{1}{2}u_1u_2^* \tag{11}$$

These satisfy Peirce's rules for the action of operators on absolutes. So a complete algebraic analog of Peirce's logic of relatives has been established. The logic of relatives and the linear associative algebra are simply related by a linear transformation. The reader will perhaps already have noted also the appearance of some concepts suggestive of quantum mechanics.

4. QUANTUM CORRESPONDENCE

We consider a quantum system describing a single free nonrelativistic particle with two internal states. Ordinarily the wave functions for this system are two-component spinors and the operators are linear combinations of the 2×2 Pauli matrices. The operators are thus numbers in the C(3, 0) Clifford algebra and also must be linear transformations of the Peirce relative operators.

There is an alternate way to represent wave functions, themselves, as 2×2 matrices instead of column or row matrices. This has the advantage that all quantities are Clifford elements and can be expressed in terms of the γ's independently of any specific matrix representation. This type of approach actually has a long and respectable history, especially with the relativistic Dirac theory, associated with names like Eddington (1928), Proca (1930), and Sommerfeld (1939). See the discussion by Snygg (1997, p. 170) for details and further references.

We adopt this approach of wave functions as square matrices. The results can all be expressed generally in terms of γ's although we will often use the specific matrix representation of the previous two sections for the sake of transparency.

A matrix representation of the momentum operator, which will appear as part of a wave function, is

$$\mathbf{p} = p_i \gamma^i = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} \tag{12}$$

where the p_i are the 3-space vector components. These are not spinors, but will be related presently to a representation of spinors by matrices.

The product

$$\mathbf{pp} = p_i p_j \gamma^i \gamma^j = \left(p_1^2 + p_2^2 + p_3^2\right) I = p^2 I \tag{13}$$

is obtained using the Clifford multiplication rule (or by squaring the matrix (12)).

The derivative operator or gradient in Cartesian coordinates is

$$\nabla = \frac{\partial}{\partial x^i} \gamma^i \tag{14}$$

with Laplacian

$$\nabla^2 = \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \gamma^i \gamma^j = \left(\frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2} \right) I \tag{15}$$

The Schroedinger equation is

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi \tag{16}$$

Now the usual assumption of a plane wave solution is made,

$$\psi = \mathbf{u}\exp\left(ip_i x^i/\hbar\right) \tag{17}$$

but with **u** a Clifford number which will be represented as a 2×2 matrix. One has

$$\nabla\psi = \frac{i}{\hbar}\mathbf{p}\psi \tag{18}$$

$$\nabla^2\psi = -\frac{p^2}{\hbar^2}\psi \tag{19}$$

which gives, using (16) and (17),

$$p^2\mathbf{u} = 2mE\mathbf{u} \tag{20}$$

The Clifford algebra can also be consistent with standard Heisenberg commutators. The algebra elements (such as the Pauli matrices) also satisfy commutation relations. One also needs to recall that the operators should be considered as acting on some Clifford matrix Φ. The position operator is

$$\mathbf{x} = x_i\gamma^i = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} \tag{21}$$

An example of a commutator is

$$\begin{aligned}
[x_1\gamma^1, p_1\gamma^1]\Phi &= (x_1\gamma^1)(p_1\gamma^1)\Phi - (p_1\gamma^1)(x_1\gamma^1)\Phi \\
&= (x_1\gamma^1)\left(\frac{\hbar}{i}\frac{\partial}{\partial x^1}\gamma^1\right)\Phi - \left(\frac{\hbar}{i}\frac{\partial}{\partial x^1}\gamma^1\right)(x_1\gamma^1)\Phi \\
&= (x_1\gamma^1)\left(\frac{\hbar}{i}\frac{\partial}{\partial x^1}\gamma^1\right)\Phi - \frac{\hbar}{i}\gamma^1\gamma^1\Phi(x_1\gamma^1)\left(\frac{\hbar}{i}\frac{\partial}{\partial x^1}\gamma^1\right)\Phi \\
&= i\hbar\Phi
\end{aligned} \tag{22}$$

where the Leibniz rule (Snygg, 1997, p. 86) for the derivative of a product of Clifford numbers has been used.

Note that the Pauli matrices are representations of generators of SU(2) and thus, in addition to the Clifford algebra rule, also satisfy the commutation relations

$$\gamma^i\gamma^j - \gamma^j\gamma^i = 2i\varepsilon^{ijk}\gamma^k \tag{23}$$

This is compatible with the Clifford multiplication rule. For example, one can write

$$\gamma^i\gamma^j = i\varepsilon^{ijk}\gamma^k \tag{24}$$

for $i \neq j$. This is just (1c).

As an example of this, consider the commutator

$$\begin{aligned}[x_3\gamma^3, p_1\gamma^1]\Phi &= (x_3\gamma^3)(p_1\gamma^1)\Phi - (p_1\gamma^1)(x_3\gamma^3)\Phi \\ &= i\gamma^2(x_3p_1 + p_1x_3)\Phi \\ &= \hbar\gamma^2\left(x_3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^1}x_3\right)\Phi \\ &= 2\hbar\gamma^2 x_3\frac{\partial}{\partial x^1}\Phi \end{aligned} \tag{25}$$

Generally,

$$[x_i\gamma^i, p_j\gamma^j]\Phi = i\hbar\delta_{ij}\Phi + 2\hbar\varepsilon^{ijk}\gamma^k x_i\frac{\partial}{\partial x^j}\Phi \tag{26}$$

where there is no sum over indices. This is the usual commutation relation plus an additional term. It seems to be a natural generalization of the Heisenberg commutators.

We now seek a suitable form for the matrix factor $\mathbf{u}$ of the wave function. It is natural to assume that $\mathbf{u}$ is an eigenfunction of the momentum operator $\mathbf{p}$ such that,

$$\mathbf{pu} = p\mathbf{u} \tag{27}$$

A simple choice for $\mathbf{u}$ which satisfies (27) is

$$\mathbf{u}_+ = pI + \mathbf{p}\begin{pmatrix} p + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p - p_3 \end{pmatrix} \tag{28}$$

That is,

$$\mathbf{pu}_+ = p\mathbf{p} + p^2 I = p(pI + \mathbf{p}) = p\mathbf{u}_+ \tag{29}$$

using (13).

We notice now that for $p_1 = p_2 = 0$, then $p_3 = p$ (motion is along the 3-axis) and

$$\mathbf{u}_+ = 2p\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 2pu_{11} \tag{30}$$

So $\mathbf{u}_+$ is directly related to the Peirce relative operator u_{11}.

A second solution of interest is

$$\mathbf{u}_- = pI - \mathbf{p} \tag{31}$$

which satisfies,

$$\mathbf{pu}_- = -p\mathbf{u}_- \tag{32}$$

and for $p_1 = p_2 = 0$,

$$\mathbf{u}_- = 2p \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 2pu_{22} \tag{33}$$

Other general results are

$$\mathbf{u}_+\mathbf{u}_+ = 2p\mathbf{u}_+ \qquad \mathbf{u}_-\mathbf{u}_- = 2p\mathbf{u}_- \tag{34a}$$

$$\mathbf{u}_+\gamma^i\mathbf{u}_+ = 2p_i\mathbf{u}_+ \qquad \mathbf{u}_-\gamma^i\mathbf{u}_- = 2p_i\mathbf{u}_- \tag{34b}$$

showing that the $\mathbf{u}$'s are their own eigenmatrices and also can be idempotent with suitable normalization. In addition,

$$\mathbf{u}_+\mathbf{u}_- = 0 \tag{35}$$

The two states are orthogonal. The two $\mathbf{u}$'s are also Hermitian matrices, so $\mathbf{u}^*\mathbf{u} = \mathbf{uu}$ and each (34) is actually a matrix of expectation values.

We note that, using the projectors $u_{11} = (1/2)(I + \gamma^3)$ and $u_{22} = (1/2)(I - \gamma^3)$ from (7) or (5) acting on the right, one obtains two solutions which are just matrices with one nonzero column. These are left ideals of the algebra and correspond to spinor solutions.

An helicity operator can be defined,

$$\mathbf{h} = \frac{\mathbf{p}}{p} \tag{36}$$

so that

$$\mathbf{hu}_+ = \mathbf{u}_+ \qquad \mathbf{hu}_- = -\mathbf{u}_- \tag{37}$$

These are obviously spinup and spindown states.

A Hamiltonian operator is one which gives $\mathbf{Hu} = E\mathbf{u}$ for either $\mathbf{u}_+$ or $\mathbf{u}_-$. There are many possible forms for $\mathbf{H}$. We choose

$$\mathbf{H} = \frac{1}{8m}\left(\mathbf{u}_+\mathbf{u}_+^* + \mathbf{u}_-\mathbf{u}_-^*\right) \tag{38}$$

The $\mathbf{u}$'s are Hermitian, but the operator is written explicitly with $\mathbf{uu}^*$ since this suggests the ket-bra notation $|\rangle\langle|$.

It is easy to verify, for example, that

$$\mathbf{Hu}_+ = \frac{p}{4m}\mathbf{u}_+\mathbf{u}_+ = \frac{p^2}{2m}\mathbf{u}_+ = E\mathbf{u}_+ \tag{39}$$

using (34a).

We have established how the quantum eigenstates $\mathbf{u}_+$ and $\mathbf{u}_-$ are related to the Peirce relative operators u_{11} and u_{22}. A quantum correspondence can also be established for the operators u_{12} and u_{21} and the Peirce absolutes u_1 and u_2.

To accomplish this, we make use of the matrix γ^1 (see (1b)) and note that if $\mathbf{w}$ is a Clifford number and $\mathbf{u}$ a Schroedinger solution then $\mathbf{uw}$ is also a solution.

In particular, $\mathbf{u}_+\gamma^1$ and $\mathbf{u}_-\gamma^1$ are solutions and correspond (for momentum along the 3-axis) since

$$\mathbf{u}_+\gamma^1 = 2p\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = 2pu_{12} \tag{40a}$$

to a spinup state and since

$$\mathbf{u}_-\gamma^1 = 2p\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = 2pu_{21} \tag{40b}$$

to a spindown state.

Further, one can construct general solutions:

$$\mathbf{U}_+ = \mathbf{u}_+ + \mathbf{u}_+\gamma^1 \tag{41}$$

This has the correspondence

$$\mathbf{U}_+ = 2p\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} = 2pu_1 \tag{42}$$

for $p_1 = p_2 = 0$, giving a direct quantum interpretation of the Peirce absolute. Equations (41) and (42) correspond to (9a). A similar comparison holds for spindown states (9b). The Peirce absolutes, u_1 and u_2, represent matrix compilations of spinup and spindown states, respectively.

It is not difficult to compute (using (34)) general results such as

$$\mathbf{U}_+\mathbf{U}_+ = 2(p + p_1)\mathbf{U}_+ \tag{43}$$

Also, since

$$\mathbf{U}_+^* = \mathbf{u}_+ + \gamma^1\mathbf{u}_+ \tag{44}$$

the **U** states are not Hermitian, but the general result

$$\mathbf{U}_+\mathbf{U}_+ = 4(p + p_1)\mathbf{u}_+ \tag{45}$$

parallels the first equation in (11). Analogs of the rest of (11) can also be obtained.

The point is, one can start with two mutually exclusive absolutes, spinup u_1 and spindown u_2 in this case. From these states the relative operators (11) can be formed which are themselves factors of wave function solutions. The canonical wave function factors $\mathbf{u}_+$ and $\mathbf{u}_-$ then appear as elements of the Peirce/Clifford algebra. A well-known quantum theory can thus be produced with only two Peirce absolutes as a basis. It should be remarked that the quantum superposition principle applies since any linear combination of solutions is, in general, a mixed state which is also a solution.

In addition to the free-particle theory just developed, interacting systems can also be described. A general operator can be formed from single particle states:

$$A = a_{11}u_{11} + a_{12}u_{12} + a_{21}u_{21} + a_{22}u_{22} \tag{46}$$

where the a_{ij}'s are complex coefficients.

A transition from an initial state ψ_i to a final state ψ_f is just

$$\psi_f = A\psi_i \tag{47}$$

that is, a multiplication of Clifford numbers.

So the interaction of one particle with another could be given by (47) with A a linear combination of states like (46) which could be formed of operators associated with a second particle. So quantum interactions, at least schematically, can be adapted to the Peirce formalism. A diagrammatic approach is outlined in the next section.

There is a relationship here, which needs to be explored further, between the present theory and a newly proposed theory of Clifford statistics (Baugh et al., 2001; Finkelstein and Galiautdinov, 2001). In the case of "Cliffordons" symmetries of multiparticle states arise from the permutation group. A swap of two Clifford states is produced by a sum or difference of two fundamental operators. For example, a special case of (46):

$$S = \mathbf{u}_{12} + \mathbf{u}_{21} \tag{48}$$

with

$$S\mathbf{U}_+ = \mathbf{U}_- \quad S\mathbf{U}_- = \mathbf{U}_+ \tag{49}$$

So S generates transitions between spinup and spindown states.

5. DIAGRAMS

Peirce invented several systems of graphs or diagrams to model his various logical constructs. We will not attempt to enumerate them here since they are both extensive and sophisticated. Significantly, however, we have been unable to discover in his writings any overt instance in which he identified the types of diagrams which might apply to the logic of relatives and its associative algebra. This omission is surprising considering that both diagrams and algebras occupied a large part of his time.

The omission is also surprising since there is, indeed, a set of diagrams which does correspond to the algebra of the logic of relations. These diagrams are the beta existential graphs (see, Peirce, 1933, paragraph 468), in particular, the triadic subset of these graphs.

A general triadic graph is just a central vertex from which three lines radiate. It simply says that three things are related by some fact. An example of a triadic relation would be, "Susan sold her car to Ike." A triadic relation is logically distinct from a dyadic relation such as "George is greater than Sam." Peirce proposed a reduction principle which states that triads cannot be composed of dyads. This was recently proved by Burch (1991) using rigorous modern logic.

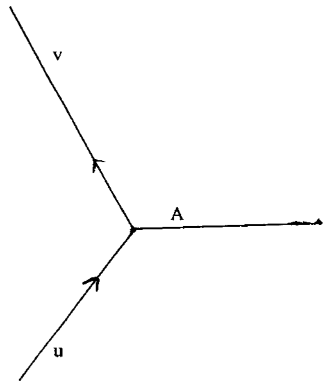

Fig. 1. Basic triadic graph.

Beta graphs are bonded together by joining one line to another in pairs only. Thus a network of any extent can be formed which is composed only of triads.

A triadic graph, then, of the algebraic expression $Au = v$ or, verbally, "A operating on u gives v" would be that of Fig. 1. Here, u represents an initial state of a single particle, A is an operator representing an interaction with an external particle or field, and v is the final state of the particle. The u and v lines might refer to the wave functions of single particle states, $|u\rangle$ and $|v\rangle$, the A line to an operator $|v\rangle\langle u|$. The wave functions are assumed to be normalized. This implies integration over appropriate spaces according to common practice.

Two operations in succession would give C = BA as shown in Fig. 2. This figure shows a Peirce joining of the lines representing the intermediate v state. Two triads are thus bonded into a single tetrad graph. A further joining of the interaction lines A and B to represent a single effective interaction C produces a new triad.

This could be the history of the states of a particle as produced by successive interactions.

The interaction of two particles could be depicted as in Fig. 3.

The interaction A changes the state of one particle from u_1 to u_2 and the other from v_1 to v_2. The operator A could be composed of states either of the u particle or of the v particle. The joining here is of lines representing interactions.

Figure 3 looks somewhat like a Feynman diagram. It actually represents a sum over all orders of Feynman diagram. It is more of the nature of an S-matrix diagram in a Born approximation. We will take it to be a schematic of a two-particle interaction, but with specific algebraic implications. That is, given the Peirce/Clifford functions u, v, and A the actual matrix wave functions $u_2 = Au_1$ and $v_2 = Av_1$ can be computed as multiplication of Clifford algebra matrices. An explicit justification of this will be given in future work.

Diagrams representing any number of interconnected particle interactions can be formed using the state joining of Fig. 2 and the interaction joining of Fig. 3. One

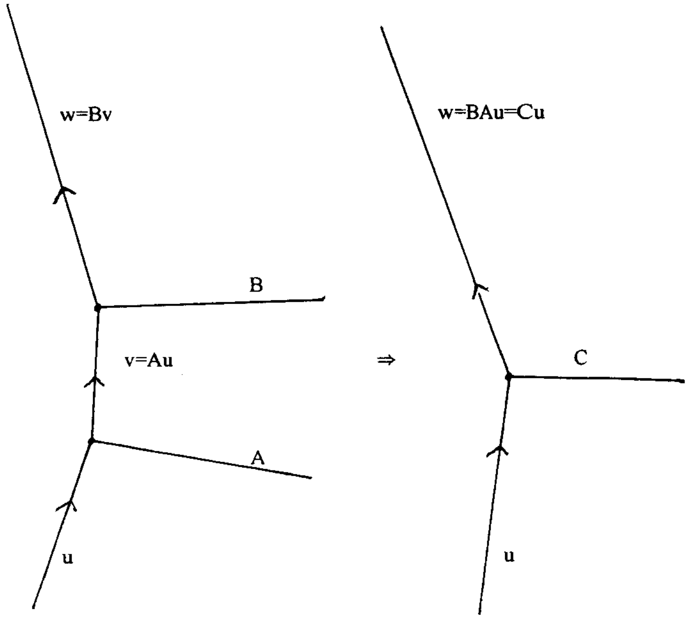

Fig. 2. Graphical representation of operator multiplication (C = BA).

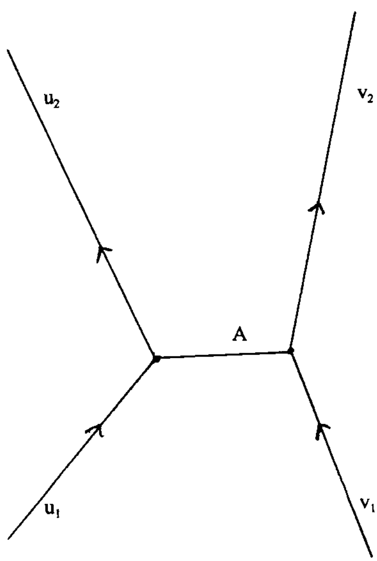

Fig. 3. Graph of particle interaction.

could imagine a network of triads continuously joining in space and time forming a web which could encompass the universe. A similar idea was once proposed by J. A. Wheeler (Wheeler, 1983).

One could also envision such diagrams as a way of planning the successive interactions of elementary particles for various experiments. For example, it might be possible to simulate basic computer functions such as logic gates using the interconnection of several interactions such as Fig. 3.

Very recently, Mermin (2001) has proposed a graphical scheme which is somewhat similar to ours. His is based on a certain type of logical gate (controlled-NOT), whereas our diagrams include general particle interactions as Clifford products.

6. DISCUSSION

We have been able to make a plausible sketch of a path from Peirce and Clifford in the 1870s to the quantum mechanics of the 1920s. Of course, it is a road not taken. They were 50 years ahead of their time. One can only speculate what might have happened if Peirce's life had been less chaotic and Clifford's life longer.

The ideas developed here, however, including the use of Clifford matrix wave functions, Peirce's relatives as quantum operators, the generalized commutators, and the triadic diagrams, might provide some useful insights even at this late date. In particular, work is progressing on a relativistic theory using Dirac matrix wave functions.

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