Abstract

We offer a formulation of first-order logic that is diagrammatic in nature, retains a basic part of C. S. Peirce's own graphics, renders quantification both graphical and easily readable, and seems a natural approach to what lies at or near the foundations of his science of Semeiotic. The approach is in agreement with Peirce's propositional graphics (Alpha EG), but different in its treatment of quantifiers. Our syntax is formulated with the "atoms" of the usual predicate calculus given a fine structure in terms of a bonding operation on monadic, dyadic, and triadic building blocks. A suitable semantics is introduced, based on relative product and culminating with a proof of a precise formulation of the famous Peircean Non-Reduction Thesis (**NRT**). (In particular, in Theorem **B'**, we document precisely the quantificational aspect of **NRT**: quantification is a multivalent operation, with full expressivity occurring only for valencies ≥ 3.)The principles **NRT** and **RCT** (Relational Completeness Thesis) are implicit in predicate logic, whatever form it might take; but their explicit presentation has generally been either neglected (classical predicate calculus) or attended by technical complexities (Peirce's Beta EG, Burch's PAL). Here, the proofs are both explicit and relatively simple. Other contexts in which our "molecular" treatment of relations may be advantageous will be indicated.

Keywords: Charles Peirce, diagrammatic notation, relational bonding, quantifier valency, orbits, non-reduction theorem, Semeiotic.

"Betagraphic": An Alternative Formulation of Predicate Calculus

INTERDISCIPLINARY SEMINAR ON PEIRCE[1]

Preface

There are at least a few plausible grounds for our use of the term *Beta* in our title, notwithstanding that there is a key departure, in our framework, from classical Beta Existential Graphs (β EG). The situation, in brief, is as follows.

The reader accustomed to Peirce's graphical development of quantificational logic may, if desired, continue to think of formulas ("betagraphs") being written on a "sheet of assertion." We retain the "cut" notation (a surrounding oval) for negation and continue to represent conjunction simply by juxtaposition of diagrams. However, on the side of departure, we dispense with the sometimes problematic "lines of identity" of classical Beta and instead represent a quantification by a "bouquet" of lines leading from a point outside the "open" (i.e., quantifier-free) part of the diagram inward to the various "open hooks" (= free variable spots) to which the quantifier is to apply, the external point in question being labelled by the object label (= variable) under quantification.

Occasionally an open formula (i.e., an unquantified betagraph) upon which we wish to focus special attention will be enclosed within a *rectangle* (as opposed to an *oval*, which is reserved for negation); lines of quantification, if any, will enter such formulas from outside such enclosing rectangles. (This use of an enclosing rectangle to say "Here is a formula you might wish to regard carefully" will also appear extensively in the latter part of Section Three, *as an abbreviative device.*)

We will sometimes present an example in betagraphic form and then give its standard linear-notational counterpart; the reader may then compare the two notations. It may happen that "ease of reading" is no greater in the betagraphic formulation, but we contend that there are cases in which it definitely *is* greater there; this will be particularly apparent at a certain point in Section Three. At other times, our injection of linear notation (again, we cite Section Three) is simply to save space.

A key feature of our notation is the "molecular" representation of relations, based, in close accord with Peirce's own ideas, on a bonding operation applied repeatedly to relations of arities one, two, three. We think such representation may well be of descriptive utility in a variety of contexts in which the formalism might have application. What might such uses be? One possibility currently under consideration has to do with representation of neuronal activity at various levels of microscopic detail. The study of interpersonal networking phenomena is another area in which the "molecular" approach appears to offer descriptive value. (Respecting both topics, note the concluding paragraph (4) at the end of the paper, where some remarks on Semeiotic may also be found.)

It should not be supposed that we seek to offer here a path to a clearer vision of Peirce's own graphical system Beta; on the contrary, our whole

emphasis is on providing a strongly diagrammatic approach that does not involve the intricacies of the Peircean graphs, vis-à-vis quantification (or, similarly, the rather formidable notational apparatus of Burch's algebraic system PAL (Burch 1991), which in addition to being notationally somewhat dense is, in and of itself, non-diagrammatic). In particular, although the theorems in Section Three are not, in substance, new, we believe that we have succeeded in reducing the notational and procedural complexity required for the *proofs* of these results.

Moreover, it should be noted that Geraldine Brady and Todd Trimble have recently shown, for those wishing to see how Peirce's graphs fare when operated on by the contents of a powerful present-day mathematical toolkit, that first Alpha, then Beta, and perhaps Gamma **EG** as well, can be formalized, analyzed, and characterized in category-theoretic terms. Readers wishing to look into that development should probably start with their analysis of Alpha, the purely propositional part of Peirce's graphics (Brady and Trimble 2000). As will be clear in what follows, we here aim far below that level of abstract generality, seeking simply to put together a diagrammatic form of first-order predicate logic that (*i*) retains at least the propositional format of Peirce's graphics, along with his triad-based view of relations, (*ii*) is comparatively easy to manipulate, (*iii*) permits a fairly transparent derivation of the celebrated "Reduction Thesis" in full semantic dress, and (*iv*) appears to offer descriptive utility in a variety of contexts.

Part I: Syntax

Our intent here is not to make Peirce's original system of Beta Existential Graphs (hereafter 'Beta' or 'β'), viewed as a graphical version of first-order predicate calculus, easier to read and manipulate than many people have found it to be in its recent appearances in the literature (Burch 1991, Correia 2008, and Shin 2002, for example), not to mention in its earliest form(s) in Peirce's writings. (Contrary to received opinion, he *did,* in a widely consulted location, *publish* a solid account of both Alpha and Beta EG: see Peirce 1902.) Instead, we wish to seriously modify the notation of the quantificational core of Beta without disturbing the notational form of its propositional subsystem Alpha. The resulting system remains strongly diagrammatic and not wholly "un-Peircean," yet (we believe) reasonably easy on the eye as planar diagrams go. Above all, it will be a system in which the *interpretation* of a given formula (well-formed diagram) is generally rather easy.

The ground-level relational framework of the formalism will be set up in such a way that the resulting system, when furnished with an appropriate semantics, readily exhibits the "Relational Completeness" [**RCT**] portion (see our previous paper, Interdisciplinary Seminar on Peirce 2011) of Peirce's so-called *Reduction Thesis*, as well as accommodating an appropriate version of the non-reduction [**NRT**] portion

of the thesis. (We are here using the narrow but precise sense of Peirce's term 'Reduction Thesis' as meaning **RCT** + **NRT** relative to a particular interpreted formal system.)

Of course, any formulation of such a system will become progressively more irksome to read as its expressions become progressively more complex; but we seek to arrange matters in such a way that at least its two and three-quantifier "graphs," a subclass of considerable importance, are easy to work with.

1. The "Alphabet."

1.a. We shall have lists

$$\bullet_1,\bullet_2,\bullet_3,\ldots;\ \bullet\bullet_1,\bullet\bullet_2,\bullet\bullet_3,\ldots;\ \bullet\bullet\bullet_1,\bullet\bullet\bullet_2,\bullet\bullet\bullet_3\ldots$$

of monadic, dyadic, and triadic predicate (or relation) symbols, respectively. Specific meanings will subsequently be attached to $\bullet_1$, $\bullet\bullet_1$, and $\bullet\bullet\bullet_1$, ; otherwise, all these symbols are to be regarded as *variables.*

1.b. We shall have a list $x_1, x_2, x_3, \ldots$ of unspecified (i.e., uninterpreted) *object labels* (= "individual variables"), and a list $c_1, c_2, c_3, \ldots$ of *formal constants*, the latter to be used for naming specific objects in some "universe of discourse." We shall use letters *x*, *y*, *z*, etc. to denote the presence of an (unspecified) object label, and letters *a*, *b*, *c*, etc. to denote the presence of an (unspecified) constant. (Our displayed examples will be simple enough to ensure that the front and back ends of the English alphabet do not collide.) Wherever *distinct* letters *x*, *y*, *z*, etc. (or *distinct* letters *a*, *b*, *c*, etc.) occur in a given diagram, it is to be understood that they signal the presence of *distinct* object labels (or *distinct* constants).

2. Well-Formed β-Graphs (WFBGs).

Most of the constructional and notational complication in what we will do involves our description of the lowest-level formulas for the system we will set up: these low-level formulas are what we shall term "*molecules.*" They correspond to the "atomic formulas" of a standard version of predicate calculus; here, however, we apply the term *atomic* only to monadic, dyadic, and triadic relational forms, with relational forms of higher arities (i.e., valencies) to be built up from these three types by a procedure to be called *bonding.* (Care will be taken to specify the bonding procedure in such a way that any "molecule" can be *decoded* into its constituent "atoms," with discovery of the order in which the bondings were done.)

2.a. Atomic Well-Formed β-Graphs (atomic WFBGs).
The following graphs (and only these) count as *atomic:*

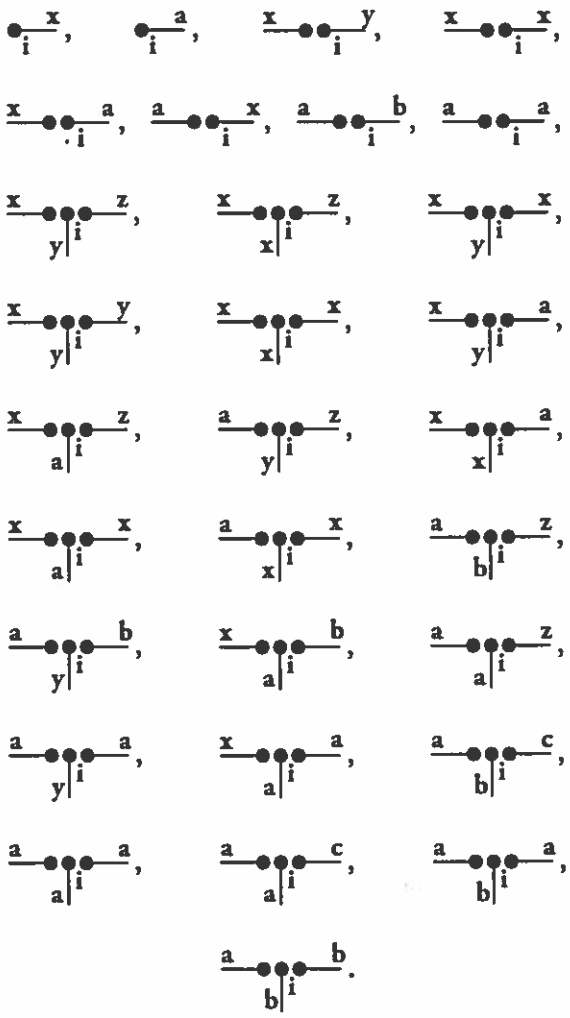


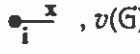
Figure 1.

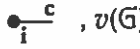
(Here, indices adjacent to bullets simply signal enumeration (i = 0, 1, 2, . . .) of individual atoms of the given form. Also, in any one of these forms, the object labels and constants are indicated generally, not specifically. So, for example,  should be understood as a shorthand reference to all the countably many triadic atoms that have distinct object labels on their horizontal hooks and some constant on the vertical hook, e.g.,  .)

In Figure 1, the lines bearing letters are referred to as "*hooks.*" A hook is *open*, and eligible for bonding, if it bears an object label (*not* a constant) and has not already been subjected to bonding or quantification (which is certainly the situation for any object-label-bearing hook

occurring in an atomic **WFBG** in Figure 1); otherwise, the hook is *closed* and ineligible for bonding.

2.b. Larger WFBGs from smaller ones, part 1: "molecules."
Let $\mathfrak{G}$ be an atomic graph. By the *valency*, $v(\mathfrak{G})$, of $\mathfrak{G}$ we mean the number of its open hooks, which can be 0, 1, 2, or 3. By the *adicity*, $a(\mathfrak{G})$, of $\mathfrak{G}$ we mean the number of mutually distinct object labels occupying its (open) hooks; again this can be 0, 1, 2, or 3. Clearly, however, it is always the case that $v(\mathfrak{G}) \geq a(\mathfrak{G})$. Some examples:

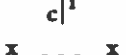
For $\mathfrak{G}=$  , $v(\mathfrak{G}) = a(\mathfrak{G}) = 1$;

For $\mathfrak{G}=$  , $v(\mathfrak{G}) = a(\mathfrak{G}) = 0$;

For $\mathfrak{G}=$  , $v(\mathfrak{G}) = a(\mathfrak{G}) = 2$;

For $\mathfrak{G}=$  , $v(\mathfrak{G}) = 2$ and $a(\mathfrak{G}) = 1$;

For $\mathfrak{G}=$  , $v(\mathfrak{G}) = a(\mathfrak{G}) = 2$;

For $\mathfrak{G}=$  , $v(\mathfrak{G}) = 3$ and $a(\mathfrak{G}) = 1$;

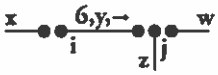
etc.

The definitions of valency and adicity will extend straightforwardly to molecules in general.

Now, in any relation, at least as far as standard logical formalisms are concerned, the related entities are arranged in a definite order: first element, second element, and so on. In the case of atomic **WFBGs**, this order can be read from the graph as going counterclockwise through the hooks starting from the one pointing to the left. As we shall soon legislate, pairs of molecular **WFBGs**, so far consisting just of atomic ones, can be *bonded* at a pair of open hooks, one from each molecule, that *bear the same object label;* and, as we shall also legislate a little further on, a hook bearing an object label may be extended to become a "line of quantification" (and thereby lose its open status).

2.b.(i). Bonding of pairs of atoms.
Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be distinct atoms. (Separate *copies* of the same atom count as a distinct pair.) Let x be an object label, and let h_i be a hook of $\mathfrak{G}_i$ that bears the label x, $i = 1, 2$. Then we may *bond* $\mathfrak{G}_i$ *to* $\mathfrak{G}_2$ *along* h_1, h_2, an operation that fuses ("welds"[2]) those two hooks into a *segment joining* $\mathfrak{G}_i$ *to* $\mathfrak{G}_2$; at the same time, we (*1*) *detach* x *from* h_i, $i = 1, 2$, and (*2*) *tag the bond segment with the notation* 6, x, $\leftrightarrow$, where the arrow points *from* $\mathfrak{G}_i$ *to* $\mathfrak{G}_2$. (We will explain shortly why we put the number 6 in our tag.)

Notice that we said we are bonding $\mathfrak{G}_1$ *to* $\mathfrak{G}_2$; this wording signals a *definite order* of operands; that order is in turn signaled by the direction of the arrow. An example:



This "says": the dyadic atom on the left, indexed *i*, is bonded to the triadic atom on the right, indexed *j*, at the common label *y*.

As we shall see, this way of doing the bonding will enable us to *reconstruct the development* of the resulting molecule to see "where it came from."

2.b. (ii). Molecules in general.

Definition 1. By a *molecule*, we mean the result of a finite sequence of bondings of distinct graphs (two separate copies of the same graph, we repeat, may be counted as distinct), starting with a pair of atoms, where each bonding occurs between hooks *occupied by the same object label*, and where each bond segment bears an appropriate tag that expresses (*a*) the point in the bonding sequence at which that particular bond occurs, (*b*) the common object label that was borne by the now-bonded hooks, and (*c*) the order of the bonding, that is, "what was bonded to what," at that point in the process.

For the sake of (*a*), we need to carefully define what we shall call the *bonding number*. Thus:

Definition 2. For any molecule $\mathfrak{G}$, we define its *bonding number*, $\#\mathfrak{G}$, as follows. If $\mathfrak{G}$ is an atom, $\#\mathfrak{G} := 1$ (' := ' means "... is defined as ..."). Now suppose $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are molecules for which $\#(\mathfrak{G}_1)$ and $\#(\mathfrak{G}_2)$ have been defined; and suppose we bond $\mathfrak{G}_1$ *to* $\mathfrak{G}_2$, which we can indicate in abbreviative notation by $\mathfrak{G}_1 \mathbf{B} \mathfrak{G}_2$. We define the bonding number $\#(\mathfrak{G}_1 \mathbf{B} \mathfrak{G}_2)$, which is the number n to be placed on the bond segment, thus: $\#(\mathfrak{G}_1 \mathbf{B} \mathfrak{G}_2) := 2^{\#\mathfrak{G}_1} 3^{\#\mathfrak{G}_2}$. (Notice that if $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are atoms, this number will be 6, consistent with what was indicated earlier.)

Definition 2 can easily be formalized as a definition of $\#(\mathfrak{G})$ by recursion. Here is an example of a two-step molecular construction, with the "reconstruction":

To reconstruct this, just look for the largest number on a bond segment; that signals the *last* bond that was made, so we break off the molecule at the tail of the arrow and then separately reconstruct both it and the one at the head of the arrow. Here is what the "reconstruction tree" for the molecule of this example looks like, where we use the symbol "**o**" to denote each node of the tree (Figure 2 should be read

from the first bondings at the bottom to subsequent bondings as one goes upwards in the diagram):

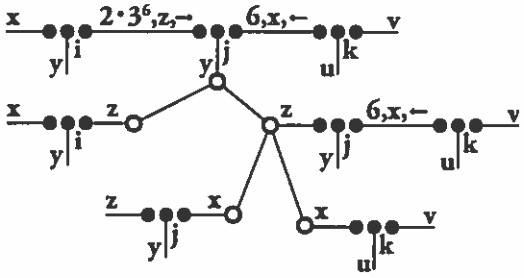
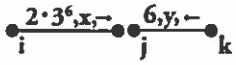


Figure 2

Next, we observe that it can happen that a sequence of bondings completely closes up all available open hooks. A simple example:

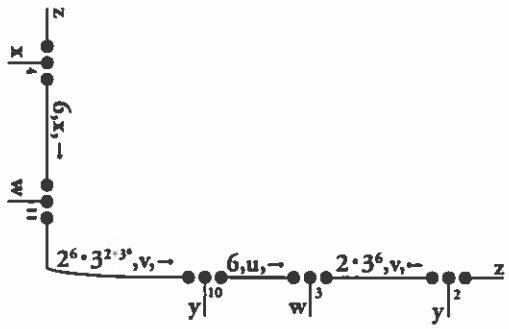


(Notice that this example illustrates the fact that we do not actually care whether the hook on a monadic atom emanates from its left or its right side.) We shall call such a graph a *nullecule* (Peirce called such graphs *medads*). When it comes time to do semantics for our formalism, in Part II, it will be explained why nullecules (with or without constants) are to be regarded as *sentences*.

Remark. If a molecule is not a nullecule, how should we *order* its hook-occupying letters? (We have already indicated an ordering convention for the letters appearing as *atoms*.) The convention that is most convenient relative to what we do in Part II (and that is by no means unnatural in its own right) is as follows: suppose molecule $\mathfrak{G}_1$ has letter-bearing hooks $h_1, \ldots, h_j, \ldots, h_k$ with its letters in that order, and molecule $\mathfrak{G}_2$ has letter-bearing hooks $k_1, \ldots, k_l, \ldots, k_m$, with its letters in that order, and $\mathfrak{G}_1$ is to be bonded to $\mathfrak{G}_2$ at the respective open hooks h_j, k_l, each of which bears the object label x. Our convention is that $\mathfrak{G}_1 \mathbf{B} \mathfrak{G}_2$ should be read with $k_1, \ldots, k_{l-1}, h_1, \ldots, h_{j-1}, h_{j+1}, \ldots, h_n, k_{l+1}, \ldots, k_m$ as the order of its letters (object labels and constants, if any). Thus we read the remaining letters of $\mathfrak{G}_1$ (if any) as being "wedged in between" the remaining letters of $\mathfrak{G}_2$ (if any). When we come to the question of *interpreting* molecules, in Part II, this convention will be seen to be tailored to our use of a certain generalization of the notion of the relative product of relations.

To conclude subsection 2b(ii), we note that there is no harm in building molecules with vertical as well as horizontal "arms," if that is found to be convenient. As an example of doing this, let us say that $\mathfrak{G}_3$ is  and $\mathfrak{G}_2 \mathbf{B} \mathfrak{G}_1$ is  , and

we wish to bond $\mathfrak{G}_3$ *to* $\mathfrak{G}_2$ **B** $\mathfrak{G}_1$. In the interest of saving space we write:



But wait: that wasn't the only possible bonding of $\mathfrak{G}_3$ to $\mathfrak{G}_2$ **B** $\mathfrak{G}_1$; we could have used the hooks bearing *z*, or those bearing *w*. We leave it to the reader to construct, as he or she sees fit, those alternative bondings. (In the case of the *w*-bonding, there is little choice but to go both vertical and "upside-down," and if a *w*-bonding were to be *followed by* a *z*-bonding, this would introduce a *loop*; however, this latter situation—first *w* then *z*—is *not allowed*: a molecule is not permitted to bond *with itself*, although it is permitted to bond with *a copy* of itself. And with a copy, we would avoid a loop.)

2c. Open formulas.

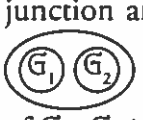
A high percentage of the price for "going graphical" with basic logical formalism has now been paid: it's clear that the "molecular" construction of higher-adicity relational frames from ones of lower adicity, specifically, from ones of adicity ≤ 3 (and in fact *valency* ≤ 3), as outlined in the preceding subsection, ultimately results in sprawling, complex, Tinkertoy-like[3] diagrams; indeed, it gets that way fairly early on in the adicity hierarchy. A simple, workable convention for abbreviating such diagrams (other than the "$\mathfrak{G}_1$ **B** $\mathfrak{G}_2$" convention, which is handy but loses far too much information) would certainly be welcome. The advantage of molecules is that they explicitly show substructure in, say, a 4-, 5-, or 6-ary relation. From here on, in any case, the syntax gets a little less rococo.

Definition 3. Conjunction. In Peirce's set-up, conjunction of graphs ("and") is indicated simply by positioning them next to each other on a pre-arranged surface known as the Sheet of Assertion. We shall follow this procedure, in that we indicate conjunction simply by juxtaposition of the **WFBGs** to be conjoined.

Definition 4. Negation. We adopt the usual practice of Peirce and his followers (it being the pictorial hallmark of Alpha, the propositional part of Beta); the negation of $\mathfrak{G}$ (i.e., "*not* $\mathfrak{G}$") shall be denoted by the inclusion of G in an oval, thus: $\textcircled{\mathfrak{G}}$.

Definition 5. The *open* **WFBGs** are the members of the smallest class containing all molecules and closed under the application of conjunction and negation.

Non-exclusive disjunction is, as usual, readily definable from conjunction and negation: given any two **WFBGs** $\mathfrak{G}_1$ and $\mathfrak{G}_2$, the graph



will count as a **WFBG** and clearly expresses that at least one of $\mathfrak{G}_1$, $\mathfrak{G}_2$ is asserted rather than denied.

We now have the "Boolean" part of our formal system and need only to add quantifiers to get the whole set-up, syntactically. (For predicate calculus, of course, we shall need to specify axiom schemes and rules of inference.)

2d. Quantifiers.

Fitting quantification in with the propositional part of logical beta-graphs in a *readily readable* way has been a problem from Peirce onward (e.g., quarrels over "lines of identity": what exactly are they? how exactly do or should they interact with negation-generated "levels," or "areas" of cuts, etc.).[4] In dealing with this issue, we shall make no attempt to preserve anyone's "original intent" or make any conjecture concerning such intent. We will instead proceed with an eye (we hope) to clarity. To start, we introduce a special notation for the *existential* quantification of an object label associated with an open hook. If the object label is x, the notation in question will be $\hat{x}$. Likewise, there will be a special symbol for the *universal* quantification of x, namely, $\check{x}$. If more than one object label (say, y in addition to x) is to be quantified in a given **WFBG**, we need to be careful about the order of quantification: existential-existential and universal-universal present no issue, but clearly, existential-universal and universal-existential must be distinguished from one another. There are at least two ways of making the distinction: first, each time we add a new quantifier, we can first double-negate the graph to be quantified and then place the quantifier symbol *outside* that double oval. Or, second, we can simply use identifying parenthetical tags on the quantified object labels. Thus, if the object label to be quantified is, say, x_2, and we are about to make a *third* quantification, say existentially for x_2, in the **WFBG** under construction, we could signal the quantification by "$\hat{x}_2(3)$," instead of simply "$\hat{x}_2$."

The objection to the first method is twofold: first, we don't particularly like double negations that function as *barriers*, like parenthesis-pairs, and sometimes cannot be deleted without destroying the well-formedness of the graph; and second, burgeoning thickets of irremovable double ovals can make a graph difficult to read, and that is the very difficulty we are trying to minimize. So we opt for the numerical tag approach: $\hat{x}\,(n)$, $\check{x}\,(n)$. Having made this choice, we can state the following rule:

Rule for forming larger WFBGs via quantification.

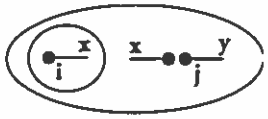
1. If $\mathfrak{G}$ is an open **WFBG**, and if for some object label x there is in $\mathfrak{G}$ at least one hook with x attached to it, then $\mathfrak{G}'$ is again a **WFBG**, where $\mathfrak{G}'$ is obtained from $\mathfrak{G}$ by (*a*) extending each such hook to a common point outside $\mathfrak{G}$ at which they (the extensions) all meet, this point then being marked with either a tag $\hat{x}(1)$ or a tag $\check{x}(1)$, and (*b*) erasing x from each such extended hook, which is now called a *line of quantification* and is declared closed. If the tag is $\hat{x}(1)$, the quantification is existential; if $\check{x}(1)$, it is universal.

2. Suppose $\mathfrak{G}$ is a **WFBG** containing quantifier tags $\hat{x}(n)$ and/or $\check{x}(n)$ with $1 \leq n \leq k$. Then, if there is in $\mathfrak{G}$ at least one open hook bearing some object label y, let the occurrences of y in $\mathfrak{G}$ be jointly quantified to produce $\mathfrak{G}'$ as in Part 1 of the rule, with the tag for this quantification being either $\hat{y}(k+1)$ or $\check{y}(k+1)$; then $\mathfrak{G}'$ is a **WFBG** with an *outermost* quantifier that is existential if the tag is $\hat{y}(k+1)$, universal if it is $\check{y}(k+1)$.

Note: Quantification reduces both valency and adicity. After quantifying the open-hook object labels in a **WFBG**, we have a graph of valency = adicity = 0; such a **WFBG** is termed a *sentence.* (As remarked earlier, *molecules* with no open hooks also count as sentences.)

Here are some simple examples illustrating use of this rule:

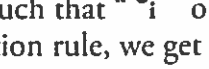
(i) Suppose $\mathfrak{G}$ is the open graph



of total molecular valency 3, adicity 2, which is designed to assert: " 

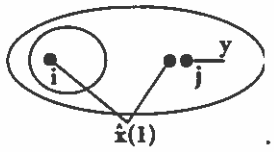
or not- 

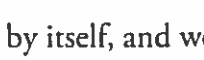
." And suppose we want to existentially quantify the object label x, to assert "there is an x such that " 

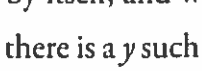
or not [

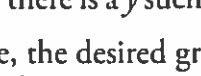
]."

Via part 1 of the quantification rule, we get the following **WFBG**:



(ii) Suppose $\mathfrak{G}$ is 

all by itself, and we wish to form from it a **WFBG** that asserts "for every x, there is a y such that 

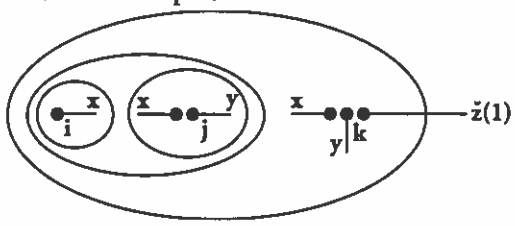
." Applying Part **2** of the quantification rule, the desired graph is, 

which seems easy enough to read.

Now, finally, we can specify the *class of all WFBGs:*

Definition 6. The class of **WFBGs** is the smallest class containing all *open* **WFBGs** and closed under negation, conjunction, and quantification.

Thus, for example,

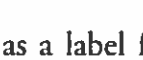
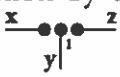


is a perfectly good **WFBG**, designed to assert: "for every *z*, it is *not* the case that we have both  *and* either or ."

3. First-order Predicate Calculus, Molecularly Based

It is now straightforward to unroll first-order logic, according to some chosen axiomatic/inferential format, in the notational framework just established. Some Peirce scholars may not like this approach; it is certainly not Peirce's approach, since he worked from his own ideas about logic and of course had no modern formulations at his disposal for comparison. However, being pragmaticists, we shall risk the complaint. Again, our primary concern is simply to have a graphical system that is easy to read and work with but not entirely "un-Peircean," a goal that naturally calls for changes of some sort.[5]

The following conventions will facilitate the discussion:

(i) The predicate symbol  is henceforth reserved as a label for *the entire domain of discourse*, in a given *universe of discourse* to which we may apply our system of **WFBGs**; the class of such universes will be formally specified in Part II of the paper. Moreover, the predicate symbol  is reserved as a label for the ordinary *binary identity relation* on any given domain. (Out of long habit, we might sometimes lapse into referring to that relation by the symbol "=".) And finally, we reserve the predicate symbol  for the so-called "teridentity" relation that would be written in standard notation as $x = y \,\&\, y = z$. In our notation, we can equivalently express *teridentity* by the **WFBG**  ; but note, in connection with later considerations, that this latter graph *has valency 4*.

(ii) By the word "*term*" we shall simply mean either an object label *x* or a formal constant *c*. (Since our alphabet contains no *function symbols*, our "terms" need be no more complicated than that.)

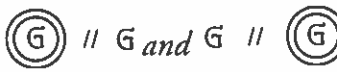
(iii) If the last step in constructing a **WFBG** G was to quantify by $\hat{x}(n)$ or $\check{x}$ (n) the *x*-labelled open hooks of a **WFBG** H, we shall denote G by $\hat{x}_{(n)}$wwwww H or $\check{x}_{(n)}$wwwww H, respectively, indicating thereby that the quantifier is applied to *all* the open-hook occurrences of *x* in H.

(iv) In stating an inference rule, if we wish to say that a **WFBG** H may be inferred from a **WFBG** G, we shall write G//H.

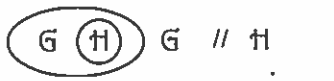
(v) To indicate that a **WFBG** 𝔗 follows from a **WFBG** 𝔊 via *our axioms and rules of inference*, we write 𝔊⇒𝔗; and to indicate that 𝔊 and 𝔗 are logically equivalent (i.e., that each so follows from the other), we write 𝔊⇔𝔗 (or 𝔗⇔𝔊).

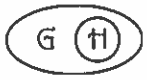
The **RULES of INFERENCE** that follow represent a slight modification (by enlargement) of the inference rules for first-order predicate logic found in Mendelson 1964, Ch. 2 § 3, where only the middle two of the following four rules appear:

R1 ("Double Negation"). For any **WFBG** 𝔊, we have[6]



R2 ("Modus Ponens"). For any two **WFBGs** 𝔊 and 𝔗,



(Observe, in connection with **R2**, that  is the form taken in our notation by the assertion "𝔊 materially implies 𝔗.")

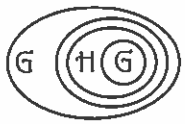
R3 ("Generalization"). For any **WFBG** 𝔊 and any object label *x* that is attached to at least one open hook in 𝔊, we have 𝔊 // x̆(n)wwwww𝔊, *n*–1 being the number of quantifiers already in 𝔊.

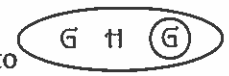
R4 ("Change of Variable"). Suppose 𝔗 is a **WFBG** arising from a **WFBG** 𝔊 by replacing an object label *x* occurring in 𝔊 at all of its occurrences[7] by a different object label *y* such that *y* has no occurrences in 𝔊. Then H//𝔊 and 𝔊//H. (And so we may write 𝔊⇔H.)

Next, the **AXIOM SCHEMAS**.

We begin with the schemata for the "Alpha part of Beta," i.e., a graphical version of propositional calculus. Again, we do not follow "classical Alpha" other than notationally; our axioms are a complete set of three as in Mendelson 1964, ch. 2.

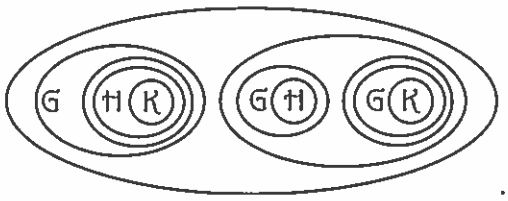
A1. For any **WFBGs** 𝔊 and 𝔗, the following is an axiom:



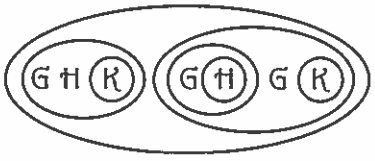
This asserts: "If 𝔊, then (if 𝔗 then 𝔊)." By **R1**, it is logically equivalent to 

, which is obviously a valid (i.e., true under all interpretations) **WFBG**.

A2. For any **WFBGs** 𝔊, ℌ, and 𝔎, the following is an axiom:

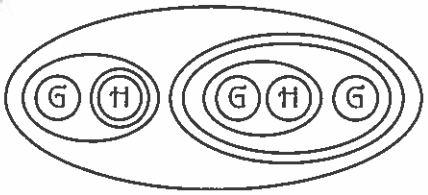


This asserts: "If (𝔊 implies (ℌ implies 𝔎)), then ((𝔊 implies ℌ) implies (𝔊 implies 𝔎))." Again using **R1**, this becomes

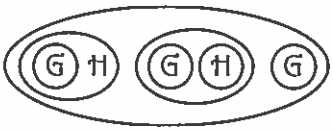


which is about the extent to which this one simplifies.[8]

A3. For any two **WFBGs** 𝔊, ℌ, the following is an axiom:



This asserts: "If not-𝔊 implies not-ℌ, then ((not-𝔊 implies ℌ) implies 𝔊)." Simplification via **R1** turns this into



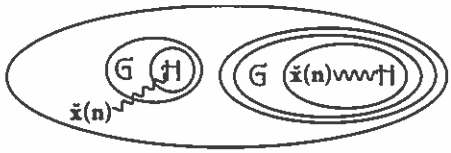
It is known that from **A1**–**A3** one can derive all tautologies, using **R1** and **R2**.[9]

The rest of the axiom base is as follows. For **A4**, we shall need the following convention: an object label *y* will be said to be *free for* an object label *x* having open-hook occurrences in a **WFBG** 𝔊, if and only if *y* = *x*, or *y* has no occurrences *of any kind* in 𝔊. (Note that this is not the way "free for *x*" is defined in standard formulations; we define it this way here on account of "molecular considerations.")

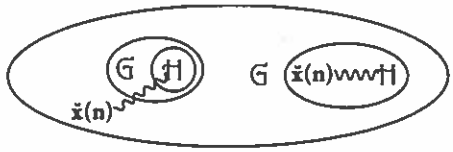
A4. Given any type of **WFBG** 𝔊, any object label *x*, and any term, *t*, where *x* has at least one open-hook occurrence in 𝔊 and where if *t* = *y* then *y* is free for *x* in 𝔊, the graph  is an axiom, where 𝔊' results from 𝔊 by replacing each such occurrence of *x* by *t*. (Note: *t* may be *x*, in which case 𝔊' = 𝔊.)

This asserts: "If for all *x* we have 𝔊, then 𝔊 holds in particular for *t*."

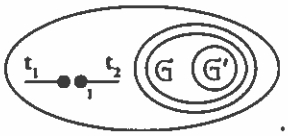
A5. If 𝔊, ℌ are **WFBGs** such that *x has no open-hook occurrences* in 𝔊, then the following **WFBG** is an axiom:[10]



This asserts "If for all x we have that 𝔊 implies ℌ, then 𝔊 implies that for all x, we have ℌ." Again, **R1** yields a bit of simplification:



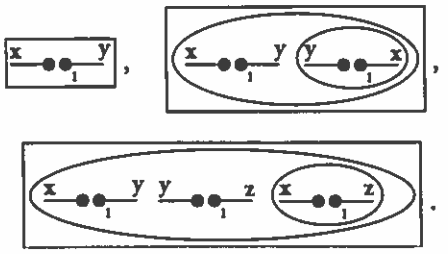
A6. Let 𝔊 be a **WFBG** containing a term t_1 that is present on at least one hook in 𝔊; and let t_2 be a second term such that t_2 is a constant if t_1 is a constant, and if t_1 and t_2 are both object labels and t_2 is different from t_1, then t_2 is free for t_1 in 𝔊. Let 𝔊' be the result of replacing t_1 by t_2 everywhere that t_1 occurs on a hook in 𝔊. Then the following **WFBG** is an axiom:



(Recalling that we have reserved  for the ordinary identity relation, **A6** simply authorizes "substitution of equals for equals.")

Finally, we include the usual equivalence-class axiom scheme governing the identity relation.

A7. The following **WFBGs** are axioms:



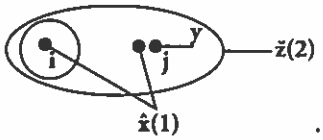
We now have what is needed for a diagrammatic treatment of first-order predicate logic that is at least superficially "Beta-istic," and it is time to attach interpretations to the syntactical skeleta. (To repeat: we have deliberately short-circuited the orthodox approach of working to get predicate logic out of Peirce's own set-up, whatever *exactly* that was; that approach is nicely exemplified in Roberts 1973 and Shin 2002.)

Postscript to Part I

(*1*) Beta graphs, as typically discussed, feature lines of identity; and these, as has often enough been pointed out, have to do with quantification.[11] Our preference here is to work explicitly with *lines of quantification*, adorned with enough notation to make their reference unmistakable. Beyond that, we have simply hijacked Peirce's notational conventions, and we have replaced the usual "atomic formulas" of standard first-order systems by "molecular" diagrams so as to give a different physical form to the syntax of predicate calculus. That predicate logic can be drawn out of "classical Beta" has been pointed out by various authors (Burch, Roberts, and Shin, to mention a few). What the preceding pages offer is nothing more than a particular way of looking at one of these systems through something like the eyes of the other.

(*2*) A word about "vacuous quantifiers." The usual predicate calculus statement of **R3**, for example, as it is found in Mendelson 1964, makes no reference to quantifying *free* occurrences of x (open-hook occurrences in Beta-istic terms); attaching a quantifier to a formula that contains nothing for it to act upon is routine in the linear-notation environment. This could be accommodated in the set-up of the present paper, but at the expense of complicating the syntax without any obvious compensatory benefits. For instance, we could enlarge the class of **WFBGs** to include such graphs as

and



.

A rule (one might call it a "rule-let") for *removing* vacuous quantifiers could be added. But our preference is simply to avoid them.

(*3*) Should bonding across negation be permitted? (I.e., should one molecule be permitted to "swim across the boundary" to join another one?) There are two ways of looking at this. First, we could interpret the result as 

; but this gives us nothing not already there if we view the bonding as preceding the negation. Or, we might construct a molecule to house the complement of whatever actual relation $\mathfrak{G}_2$ is naming in some chosen interpretation of the formalism (as we shall see in Section II can be done) and then bond $\mathfrak{G}_1$ to that molecule. That might be the better option *if we have committed ourselves to a fixed interpretation of the formalism*; but we probably don't want to make such a commitment when we are just doing syntax. So we shall agree not

to look at the matter that way. Finally, to "bond through a quantifier" would involve some rather onerous restrictions. So we shall agree (as is implicit in the preceding pages) that "bonding stops at the molecular horizon."

Part II: Semantics

A key tool for our definitions will be the operation of (generalized) *relative product*. As noted by Herzberger (to whose 1981 paper we are strongly indebted), this operation, while apparently not just acceptable to Peirce, but in fact exploited by him as a construction procedure, has one shortcoming: if the universe of discourse is finite, relative product cannot in general be used to reconstruct an $(m + n)$-ary relation from a splitting of that relation into an m-ary "left half" and an n-ary "right half," since there are in general too many elements of a relation in a finite domain for these "halves" to be tagged in a one-to-one fashion by elements of the domain. This prevents us from using the relative product procedure to verify the Relational Completeness portion of Peirce's "Reduction Thesis" in the case of a finite universe.[12] Our way of outflanking this obstacle will be to define our universes in such a way that they are all infinite, with a given "base set" (finite or infinite) extended by the set of all concatenations, $\delta_1{}^\wedge\delta_2 \ldots{}^\wedge\delta_n$, of elements (repeats allowed) of that base set. (Concatenation is a legitimate procedure for a "Peircean" semantics if relative product is, since it is in fact involved, along with *cancellation*, in the formation of relative products).

Whatever we wish to say about a finite base set $D = \{\delta_1, \ldots, \delta_k\}$ can be said within this wider context. (E.g., to say "for all x, such and such holds of x," we would simply relativize the quantifier, saying "if $x = \delta_1$ or $x = \delta_2$ or ... or $x = \delta_k$, then such-and-such holds of x," where "such-and-such" has also been relativized—if it contains any quantifiers—to $\{\delta_1, \ldots, \delta_k\}$; and of course all this can be written in our graphical formalism.) It will be shown in what follows that with this understanding about universes, our syntactic rules concerning molecular formation readily accommodate the use of relative product to produce higher-adicity relations from those of adicity ≤ 3.

Definition 7. By a *domain of discourse* we mean a set of the form $D = D_0 \cup D_0{}^\wedge$, where D_0 is a nonempty set and $D_0{}^\wedge$ is the set of all concatenations $\delta_1{}^\wedge\ldots{}^\wedge\delta_j$, $j \geq 2$, where $\{\delta_1, \ldots, \delta_j\} \subseteq D_0$.

Definition 8. Let D be a domain of discourse. By a *universe of discourse over* D we mean a sequence

$$U_D = \langle D, 2^D, 2^{D \times D}, \ldots, 2^{D \times \ldots \times D/[n \text{ times}]}, \ldots \rangle,$$

where, as usual 2^D is the set of all subsets of D, $2^{D \times D}$ is the set of all binary relations on D, and, in general, $2^{D \times \ldots \times D/[n \text{ times}]}$ is the set of all n-ary relations on D.

The next definition is, of necessity, a long one; there are many cases to be dealt with.

Definition 9.[13] Let U be a universe of discourse with domain D. (D being understood, we can write simply U, instead of U_D.) By an *atomic U-interpretation* we mean a function I_U acting on constants and atoms, and taking U-values as follows:

(*1*) $I_U(c_i)$ = some element of D, for each constant c_i;

(2)

$I_U(\bullet_1\overset{x}{\text{—}}) = \mathbf{D}$;

$I_U(\bullet_1\overset{c}{\text{—}}) = \{I_U(c)\}$;

$I_U(\bullet_i\overset{x}{\text{—}}) \in 2^D$, $i \geq 2$;

$$I_U(\bullet_i\overset{c}{\text{—}}) = \left\{\begin{matrix}\{I_U(c)\}, \text{ if } I_U(c)\in I_U(\bullet_i\overset{x}{\text{—}}) \\ \varnothing, \text{ otherwise}\end{matrix}\right\}, \ i \geq 2.$$

(*3*) [14]

$I_U(\overset{x}{\text{—}}\bullet\bullet_1\overset{y}{\text{—}}) = I_U(\overset{x}{\text{—}}\bullet\bullet_1\overset{x}{\text{—}}) = \{(\delta,\delta): \delta\in\mathbf{D}\}$;

$I_U(\overset{x}{\text{—}}\bullet\bullet_1\overset{c}{\text{—}}) = I_U(\overset{c}{\text{—}}\bullet\bullet_1\overset{x}{\text{—}}) = \{I_U(c)\}$;

$I_U(\overset{c}{\text{—}}\bullet\bullet_1\overset{c}{\text{—}}) = \{(I_U(c), I_U(c))\}$;

$I_U(\overset{c}{\text{—}}\bullet\bullet_1\overset{d}{\text{—}}) = I_U(\overset{d}{\text{—}}\bullet\bullet_1\overset{c}{\text{—}}) =$

$$\left\{\begin{matrix}\{(I_U(c),I_U(c))\}, \text{ if } I_U(c)=I_U(d) \\ \varnothing, \text{ otherwise}\end{matrix}\right\};$$

$I_U(\overset{x}{\text{—}}\bullet\bullet_i\overset{y}{\text{—}}) \in 2^{D\times D}$, $i \geq 2$;

$I_U(\overset{x}{\text{—}}\bullet\bullet_i\overset{x}{\text{—}}) = \{(\delta,\delta): (\delta,\delta) \in I_U(\overset{x}{\text{—}}\bullet\bullet_i\overset{y}{\text{—}})\}$, $i \geq 2$;

$I_U(\overset{x}{\text{—}}\bullet\bullet_i\overset{c}{\text{—}}) = \{\delta\in\mathbf{D}: (\delta,I_U(c))\in I_U(\overset{x}{\text{—}}\bullet\bullet_i\overset{y}{\text{—}})\}$, $i \geq 2$;

$I_U(\overset{c}{\text{—}}\bullet\bullet_i\overset{x}{\text{—}}) = \{\delta\in\mathbf{D}: (I_U(c),\delta)\in I_U(\overset{x}{\text{—}}\bullet\bullet_i\overset{y}{\text{—}})\}$, $i \geq 2$;

$I_U(\overset{c}{\text{—}}\bullet\bullet_i\overset{d}{\text{—}}) =$

$$\left\{\begin{matrix}\{(I_U(c), I_U(d))\}, \text{ if } (I_U(c), I_U(d))\in I_U(\overset{x}{\text{—}}\bullet\bullet_i\overset{y}{\text{—}}) \\ \varnothing, \text{ otherwise}\end{matrix}\right\},$$

$i \geq 2$;

$I_U(\overset{c}{\text{—}}\bullet\bullet_i\overset{c}{\text{—}}) =$

$$\left\{\begin{matrix}\{(I_U(c), I_U(c))\}, \text{ if } (I_U(c), I_U(c))\in I_U(\overset{x}{\text{—}}\bullet\bullet_i\overset{y}{\text{—}}) \\ \varnothing, \text{ otherwise}\end{matrix}\right\},$$

$i \geq 2$.

(4)

$I_U($  $) = I_U($  $) = I_U($  $) =$

$I_U($  $) = I_U($  $) = \{(\delta,\delta,\delta): \delta \in \mathbf{D}\}$;

$I_U($  $) = I_U($  $) = I_U($  $) =$

$I_U($  $) = I_U($  $) = I_U($  $) =$

$\{(I_U(\mathbf{c}), I_U(\mathbf{c}))\}$;

$I_U($  $) = I_U($  $) = I_U($  $) = \{I_U(\mathbf{c})\}$;

$I_U($  $) = I_U($  $) = I_U($  $) =$

$\begin{cases} \{I_U(\mathbf{c})\}, \text{ if } I_U(\mathbf{c}) = I_U(\mathbf{d}) \\ \varnothing, \text{ otherwise} \end{cases}$;

$I_U($  $) = I_U($  $) = I_U($  $) =$

$I_U($  $) = I_U($  $) =$

$\begin{cases} \{I_U(\mathbf{c})\}, \text{ if } I_U(\mathbf{c}) = I_U(\mathbf{d}) = I_U(\mathbf{e}) \\ \varnothing, \text{ otherwise} \end{cases}$,

where if *d* or *e* is missing from one of these atoms, we treat it as *c* for purposes of the valuation;

$I_U($  $) \in 2^{\mathbf{D}\times\mathbf{D}\times\mathbf{D}}$, $i \geq 2$;

$I_U($  $) = \{(\delta,\delta',\delta'): (\delta,\delta',\delta') \in I_U($  $)\}$, $i \geq 2$;

$I_U($  $) = \{(\delta,\delta',\delta): (\delta,\delta',\delta) \in I_U($  $)\}$, $i \geq 2$;

$I_U($  $) = \{(\delta,\delta,\delta): (\delta,\delta,\delta) \in I_U($  $)\}$, $i \geq 2$;

$I_U($  $) = \{(\delta,\delta'): (\delta,\delta',I_U(\mathbf{c})) \in I_U($  $)\}$, $i \geq 2$;

$I_U($  $) = \{(\delta,\delta'): (\delta,I_U(\mathbf{c}),\delta') \in I_U($  $)\}$, $i \geq 2$;

$I_U($  $) = \{(\delta,\delta'): (I_U(\mathbf{c}),\delta,\delta') \in I_U($  $)\}$, $i \geq 2$;

$I_U($  $) = \{(\delta,\delta): (\delta,\delta,I_U(\mathbf{c})) \in I_U($  $)\}$, $i \geq 2$;

$I_U($  $) = \{(\delta,\delta): (\delta,I_U(\mathbf{c}),\delta) \in I_U($  $)\}$, $i \geq 2$;

$I_U($  $) = \{(\delta,\delta): (I_U(\mathbf{c}),\delta,\delta) \in I_U($$)\}$, $i \geq 2$;

$I_U($$) = \{\delta: (\delta,I_U(\mathbf{c}),I_U(\mathbf{d})) \in I_U($$)\}$, $i \geq 2$;

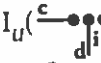
$I_U($$) = \{\delta: (I_U(\mathbf{c}),\delta,I_U(\mathbf{d})) \in I_U($$)\}$, $i \geq 2$;

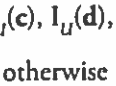
$I_U($$) = \{\delta: (I_U(\mathbf{c}),I_U(\mathbf{d}),\delta) \in I_U($$)\}$, $i \geq 2$;

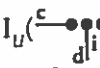
$I_U($$) = \{\delta: (\delta,I_U(\mathbf{c}),I_U(\mathbf{c})) \in I_U($$)\}$, $i \geq 2$;

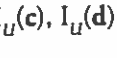
$I_U($$) = \{\delta: (I_U(\mathbf{c}),\delta,I_U(\mathbf{c})) \in I_U($$)\}$, $i \geq 2$;

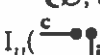
$I_U($$) = \{\delta: (I_U(\mathbf{c}),I_U(\mathbf{c}),\delta) \in I_U($$)\}$, $i \geq 2$;

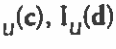
$I_U($  $) =$

$\left\{ \begin{array}{l} \{(I_U(c), I_U(d), I_U(e))\}, \text{ if } (I_U(c), I_U(d), I_U(e)) \in I_U($  $) \\ \varnothing, \text{ otherwise} \end{array} \right\}, i \geq 2;$

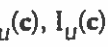
$I_U($  $) =$

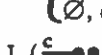
$\left\{ \begin{array}{l} \{(I_U(c), I_U(d), I_U(d))\}, \text{ if } (I_U(c), I_U(d), I_U(d)) \in I_U($  $) \\ \varnothing, \text{ otherwise} \end{array} \right\}, i \geq 2;$

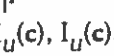
$I_U($  $) =$

$\left\{ \begin{array}{l} \{(I_U(c), I_U(d), I_U(c))\}, \text{ if } (I_U(c), I_U(d), I_U(c)) \in I_U($  $) \\ \varnothing, \text{ otherwise} \end{array} \right\}, i \geq 2;$

$I_U($  $) =$

$\left\{ \begin{array}{l} \{(I_U(c), I_U(c), I_U(d))\}, \text{ if } (I_U(c), I_U(c), I_U(d)) \in I_U($  $) \\ \varnothing, \text{ otherwise} \end{array} \right\}, i \geq 2;$

$I_U($  $) =$

$\left\{ \begin{array}{l} \{(I_U(c), I_U(c), I_U(c))\}, \text{ if } (I_U(c), I_U(c), I_U(c)) \in I_U($  $) \\ \varnothing, \text{ otherwise} \end{array} \right\}, i \geq 2.$

That concludes (mercifully) Definition 9.

We now proceed to the definition of that general form of "relative product" which we shall use for the extension of I_U to molecules. (There are other options for the notion of "generalized relative product": it depends on how one wants to *order* the object labels and constants under molecular bonding. The definition we are about to give conforms to the ordering convention for bonding laid down in Part I.)

Definition 10. Let $R(x_1, ..., x_j, ..., x_n)$ and $S(y_1, ..., y_k, ..., y_m)$ be relations with $m, n \geq 1$, $1 \leq j \leq n$, and $1 \leq k \leq m$. (We emphasize that $j = 1$, $j = n$, $k = 1$, and $k = m$ are all allowable.) The *generalized relative product* (GRP) *of R with S* (order of factors matters here) denoted by $R \bigstar_{j,k} S$, is defined to be the relation $\{(s_1, ..., s_{k-1}, r_1, ... r_{j-1}, r_{j+1}, ..., r_n, s_{k+1}, ..., s_m) : \exists l[(r_1, ..., r_{j-1}, l, r_{j+1}, ..., r_n) \in R \ \& \ (s_1, ..., s_{k-1}, l, s_{k+1}, ..., s_m) \in S]\}$.[15]

Note: if $j = n$, and $k = 1$, this is the standard basic definition of relative product.

Definition 10 is now our device for extending I_U from atoms to molecules: GRP does the job.

Definition 11. I_U for molecules. We proceed by induction on the number $b(\mathcal{M})$ of bondings used to construct the molecule $\mathcal{M}$. (Note that $b(\mathcal{M})$, the number of bonds, is different from $\#(\mathcal{M})$, the bonding number defined in Part I.) If $b(\mathcal{M}) = 0$, i.e., if the graph is an atom, the value of I_U on $\mathcal{M}$ has been prescribed in Definition 9. So suppose

$b(\mathcal{M}_0) = k + 1$; and suppose we have defined $I_U(\mathcal{M})$ for all $\mathcal{M}$ such that $b(\mathcal{M}) \leq k$. Now suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ to be molecules such that $\max(b(\mathcal{M}_1), b(\mathcal{M}_2)) = k$; let $\mathcal{M}_1 = \mathcal{M}_{11}^{\ \ x}$ and $\mathcal{M}_2 = {}^{x}\!\!-\!\!\mathcal{M}_{21}$; and let $\mathcal{M}_0 = \mathcal{M}_1 \mathbin{\mathbf{B}} \mathcal{M}_2 = \mathcal{M}_{11}^{\ \ \mathbf{n},\mathbf{x},\rightarrow}\mathcal{M}_{21}$, where $n = \#(\mathcal{M}_0)$. We define: $I_U(\mathcal{M}_0) := I_U(\mathcal{M}_1) \bigstar_{j,k} I_U(\mathcal{M}_2)$, where the bonded x-bearing hooks are at the j-th position in $\mathcal{M}_1$ and the k-th position in $\mathcal{M}_2$.[16]

An answer to the question "What happens to *constants* when we do this?" is in order. A careful examination of Definition 9 shows that in assigning I_U-values to atoms that are eligible for bonding (i.e., that have one or more open hooks), we take into account the effect of constant-bearing hooks relative to the I_U-assignment for the full-adicity atom of the same group (dyads or triads) but assign a relation of arity = the *valency* of the atom in question, maintaining respect for the order of labels. The application of GRP vis-à-vis bonding then propagates this "effect of constants" through the molecular hierarchy.

Definition 12. A *sentence* is a **WFBG** all of whose labeled hooks (if any) are occupied by constants.

Note: We remarked in Part I that nullecules, in particular, are to be regarded as sentences. In the case of *atomic* nullecules, the reading of them as sentences is pretty clear: these are atoms whose every hook is occupied by a constant. Any one of them, read as a sentence, should say that its constants, in their prescribed order, form a tuple that satisfies the relation that the atom represents, under whatever interpretation I_U is under consideration; and with respect to that I_U, this claim may be true or it may be false. With respect to *bonded* nullecules, the situation is slightly different. How could two molecules bond to form a nullecule? Clearly, they would have to bond along hooks bearing an object label x, where that hook is, for each of the two items to be bonded, *the one and only remaining open hook*. Thus, given our GRP procedure for interpretation of molecules, it must be that each of the two molecules being bonded *describes a set*, namely, some subset of whatever domain D we are looking at. On the one hand, our GRP assignment to the *bonded* molecule, i.e., the nullecule, is automatically ø.[17] On the other hand, there may or may not be a *common element* of the two sets in question, which would yield a nonempty GRP if either of the two items to be bonded had another open hook. Accordingly, we are motivated to ascribe the following sentential significance to the nullecule $M_1 \mathbin{\mathbf{B}} M_2$: $I_U(M_1) \cap I_U(M_2) \neq$ ø. (Again, for a given M_1, M_2 and I_U, this may be either true or false.)

The *truth conditions* for sentences, relative to a given U-interpretation I_U, now consist in

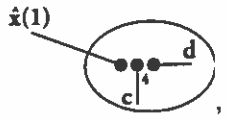
(1) the conditions for atoms laid down in Definition 9,

(2) the stipulation made regarding non-atomic nullecules in the preceding note, and

(3) for **WFBG** sentences formed using conjunction, negation, and quantification, the usual Tarski truth conditions that can be found discussed in detail in any standard text on the basics of mathematical logic (e.g., Enderton 1972 or Mendelson 1964).

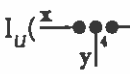
For example: If $\mathfrak{G}$ = 

, then $\mathfrak{G}$ is *true in* U *relative to* I_U if $I_U(c) = I_U(d)$, and is false in U relative to I_U otherwise. (*Notations:* "$U_{I_U} \models \mathfrak{G}$" for the true case, "$U_{I_U} \not\models \mathfrak{G}$" for the false.) Again, if $\mathfrak{G}$ =



,

then $U_{I_U} \models \mathfrak{G}$ if and only if there is an element *d* of D such that (d, $I_U(c)$, $I_U(d)$) is *not* in the relation $R \in 2^{D \times D \times D}$ for which

$I_U($ 

$) = \mathbf{R}$.

Part III: Three Theorems

We can now establish a precise form of the "Peircean Non-Reduction Thesis" (**NRT**) for a fairly wide class of formulas. Moreover, we shall verify the other half of the classical Peirce thesis, namely, the "Relational Completeness Theorem" (**RCT**), for the atoms, a result that is in Herzberger 1981 and Interdisciplinary Seminar on Peirce 2011, though of course not in the form and framework of the present paper. We begin with **RCT**.

Theorem A (RCT).

Let U be any universe of discourse, and let R be any *n*-ary relation on the domain D of U, $n \geq 4$: $R \in 2^{D \times \ldots \times D/[n \text{ times}]}$. Then there exist a U-interpretation I_U and a molecule M of valency = adicity = *n* such that $I_U(M) = R$.

(Note: since all molecules are constructed from atoms via bonding, and all atoms are at most triadic, **RCT** is clearly a statement of "Peircean relational completeness.")

Proof of Theorem A. We shall give the proof explicitly for the cases $n = 4$ and $n = 5$; the handling of cases $n \geq 6$ should then be entirely clear. So suppose, first, that we wish to obtain R, where $R \in 2^{D \times D \times D \times D}$, by molecular construction. D being infinite, there are at least as many elements in D as there are 4-tuples in R. Therefore, following the lead of Herzberger 1981, we can establish a 1–1 correspondence $\rho_i \leftrightarrow \delta_i$ between the 4-tuples $\rho_i = (r_{i1}, r_{i2}, r_{i3}, r_{i4})$ of R and certain elements δ_i of D, where *i* ranges over an index set I of cardinality card(I) = card(R) ≤ card(D). Break each element ρ_i of R into its "left half," $\{(r_{i1}, r_{i2})\}_{i \in I}$,

and its "right half" $\{(r_{i3}, r_{i4})\}_{i \in I}$. Select a pair  and  of triadic atoms, $1 < k_1 < k_2$, and choose any U-interpretation I_U such that

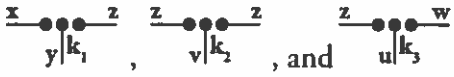
$I_U($$) = \{(r_{i1},r_{i2},\delta_i)\}_{i\in I}$ and $I_U($  $) = \{(\delta_i,r_{i3},r_{i4})\}_{i\in I}$.

Then, clearly, we have

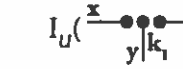
$I_U($  $) =$

$(\{(r_{i1},r_{i2},\delta_i)\}_{i\in I}) \star_{3,1} (\{(\delta_i,r_{i3},r_{i4})\}) = \mathbf{R}$

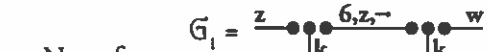
To deal in a similar way with $R \in 2^{D \times D \times D \times D \times D}$, we merely need to insert a triadic atom  in between the sort of pair we used for the 4-valent construction, split our 5-valent relation into a left side, a middle singleton, and a right side, and proceed as before. In detail: Choose



with $k_1 < k_2 < k_3$; and, letting $\tau_i \leftrightarrow \delta_i$ be a 1–1 correspondence between $\{\delta_i : i \in I\}$ and the 5-tuples $\tau_i = (r_{i1}, r_{i2}, r_{i3}, r_{i4}, r_{i5})$ of R, split each τ_i into a "left piece" $\{(r_{i1}, r_{i2})\}_{i \in I}$, a "middle piece" $\{(r_{i3})\}_{i \in I}$, and a "right piece" $\{(r_{i4}, r_{i5})\}_{i \in I}$. First, choose an I_U so that

$I_U($$) = \{(r_{i1},r_{i2},\delta_i)\}_{i\in I}$, $I_U($  $) = \{(\delta_i,r_{i3},\delta_i)\}_{i\in I}$,

and $I_U($$) = \{(\delta_i,r_{i4},r_{i5})\}_{i\in I}$.

Next, form: $\mathfrak{G}_1 =$  ,

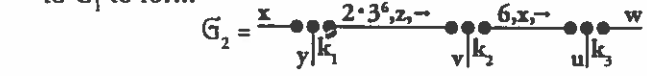
and observe that

$I_U(\mathfrak{G}_1) = \{(\delta_i,r_{i3},\delta_i)\}_{i\in I} \star_{3,1} \{(\delta_i,r_{i4},r_{i5})\}_{i\in I} =$

$\{(\delta_i,r_{i3},r_{i4},r_{i5})\}_{i\in I}$.

Finally, bond

to $\mathfrak{G}_1$ to form

$\mathfrak{G}_2 =$  ,

and observe that

$I_U(\mathfrak{G}_2) = \{(r_{i1},r_{i2},\delta_i)\}_{i\in I} \star_{3,1} \{(\delta_i,r_{i3},r_{i4},r_{i5})\}_{i\in I} =$

$\{(r_{i1},r_{i2},r_{i3},r_{i4},r_{i5})\}_{i\in I} = \mathbf{R}$.

The extension of this procedure to arities ≥ 6 is now clear: we just keep inserting additional atoms of the form  , α being some new object letter for each such insertion. Theorem A follows. □

Remark. We did not say, in the above proof, how in a given construction the function I_U should be defined *on constants*. That is because *it doesn't matter*: make that part of the definition of I_U any way you please; the argument is independent of it. With a careful choice and indexing of sequences of pairs, triples, quadruples, etc., of triadic atoms, we can in fact arrange for $2^{card(D)}$ different instances of I_U *each of which* is compatible with a construction-from-triads, using our carefully chosen and indexed atoms, of all the members of a given countable set of Rs of arities ≥ 4.

Theorem B (NRT for 3-valent open WFBGs).
(1) It is not possible to construct a molecule of valency ≥ 3 (and hence not one of adicity ≥ 3) using only monadic and dyadic atoms. As a result, there is no universe of discourse U, interpretation function I_U, and molecule $\mathfrak{G}$ constructed using only monads and dyads, such that $I_U(\mathfrak{G})$ = R, R a relation of arity ≥ 3.

(2) Let the *teridentity* or 3-diagonal relation $\{(d, d, d): d \in D\}$ on a domain D be denoted by the more compact label "*Ter*"; and let "" denote its complement in D, i.e., := D x D x D*Ter*. Further, let $\mathfrak{G}$ be an open **WFBG** that contains only molecules of valency ≤ 2, and with $a(\mathfrak{G}) = v(\mathfrak{G}) = 3$. Then $\mathfrak{G}$ cannot express either *Ter* or $\widetilde{Ter}$ or in a domain having at least 2 elements.

Proof. For (1): Given a **WFBG** $\mathfrak{G}$, we denote by $a(\mathfrak{G})$ the adicity of $\mathfrak{G}$, and recall that $v(\mathfrak{G}) \geq a(\mathfrak{G})$ for all $\mathfrak{G}$, $v(\mathfrak{G})$ the valency of $\mathfrak{G}$. If $\mathfrak{G}_1$, $\mathfrak{G}_2$ are any two molecules, it is rather obvious (and routine to verify formally by induction on the number of bondings) that if $\mathfrak{G}_1 \overset{n,x,\rightharpoonup}{\text{---}} \mathfrak{G}_2$ is a bonding of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ then $v(\mathfrak{G}_1 \overset{n,x,\rightharpoonup}{\text{---}} \mathfrak{G}_2) = v(\mathfrak{G}_1)+v(\mathfrak{G}_2)-2$. Consequently, if both $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are of valency ≤ 2, then so also is $v(\mathfrak{G}_1 \overset{n,x,\rightharpoonup}{\text{---}} \mathfrak{G}_2) \leq 2$, and this clearly implies *(1)*.

For *(2)*: We proceed by a straightforward induction on the *constructional complexity* of open **WFBGs**, which we define as follows:

$$\gamma(\mathfrak{G}) := \begin{cases} 1, \text{ if } \mathfrak{G} \text{ is a molecule;} \\ \gamma(\mathfrak{H})+1, \text{ if } \mathfrak{G} = \textcircled{\mathfrak{H}}; \\ \max(\gamma(\mathfrak{H}_1),\gamma(\mathfrak{H}_2))+1, \text{ if } \mathfrak{G} = \boxed{\mathfrak{H}_1\ \mathfrak{H}_2} \end{cases}$$

Base step: $\gamma(\mathfrak{G}) = 1$. Here $\mathfrak{G}$ is a molecule, and it is obvious that $I_U(\mathfrak{G})$ cannot be a 3-adic relation since $v(\mathfrak{G}) \leq 2$. So $\mathfrak{G}$ cannot even be a *candidate* to express *Ter*.

Now assume that we know that for $\gamma(G) \leq k$, G is not a counterexample to *(2)*.

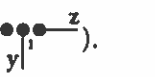
Induction step: $\gamma(\mathfrak{G}) = k + 1$.

Case 1: $\mathfrak{G} = \boxed{\mathfrak{H}_1\ \mathfrak{H}_2}$.

There are two subcases.

Subcase 1a. One of $\mathfrak{H}_1$, $\mathfrak{H}_2$ is a sentence true[18] in U, and the other has its I_U-value = *Ter* or $\tilde{T}er$. But since both $\gamma(\mathfrak{H}_1)$ and $\gamma(\mathfrak{H}_2)$ are $\leq k$, it follows from the induction hypothesis that this case presents no threat, since whichever of $\mathfrak{H}_1$, $\mathfrak{H}_2$ is *not* a sentence would have to satisfy $\nu(\mathfrak{H}_i) = a(\mathfrak{H}_i) = 3$, $i = 1$ or 2.

Subcase 1b. Otherwise. Then neither $\mathfrak{H}_1$ nor $\mathfrak{H}_2$ can have valency 3, since otherwise we would be in Subcase 1a. The valency of $\mathfrak{G}$ must therefore be split between $\mathfrak{H}_1$ and $\mathfrak{H}_2$; say, $\nu(\mathfrak{H}_1) = 2$ and $\nu(\mathfrak{H}_2) = 1$. Now suppose that we have

$I_U(\mathfrak{H}_1\ \mathfrak{H}_2) = Ter = I_U($  $)$.

Recalling that card(D) ≥ 2, let δ_1, δ_2 be distinct elements of D; and notice that $I_U(\mathfrak{H}_1)$ must be the ordinary binary identity relation on D, while $I_U(\mathfrak{H}_2)$ is simply some subset of D. Clearly, it cannot be the case that $I_U(\mathfrak{H}_2) = \emptyset$. But neither can $I_U(\mathfrak{H}_2)$ be *non*-empty; for suppose, e.g., that $\delta_1 \in I_U(\mathfrak{H}_2)$. Then $(\delta_2\ \delta_2,\ \delta_1) \in I_U(\mathfrak{H}_1\ \mathfrak{H}_2)$, with $\delta_1 \neq \delta_2$. Thus, again, $\boxed{\mathfrak{H}_1\ \mathfrak{H}_2}$ cannot in fact have *Ter* as its I_U-value. The argument is entirely similar if we assume $I_U(\mathfrak{G}) =$, except that there we would observe that $I_U(\mathfrak{H}_1)$ would have to be all of D x D.

Case 2: $\mathfrak{G} = (\mathfrak{H})$. Suppose it were the case that

$I_U(\mathfrak{G}) = Ter = I_U($ $)$. Then we would have

$I_U((\mathfrak{G})) = I_U(((\mathfrak{H}))) = I_U(\mathfrak{H}) = \tilde{T}er$. But $\gamma(\mathfrak{H}) = k$, and this contradicts the induction hypothesis; so again, it cannot in fact be the case that $I_U(\mathfrak{G}) = Ter$. Similarly if we assume $I_U(\mathfrak{G}) = \tilde{T}er$, and the proof is complete. □

Theorem B' (NRT for Quantifiers)

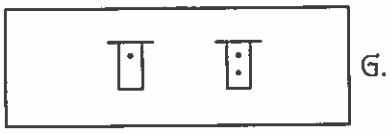
What about quantifiers? Our final theorem deals with that question, in that we shall argue that Theorem **B** almost holds for **WFBGs** that are quantified but in which no quantifier acts on more than two occurrences of its target object label. In what sense are we using the qualifier "almost"? In Theorem **B**, the only demand we made on our "universe" was the very minimal requirement that card(D) ≥ 2. Here we find it expedient to return to our normal semantic environment featuring a full universe of discourse in which the domain D satisfies card(D) ≥ $\aleph_0$.

Further, we cannot apply the same straightforward inductive strategy as in our proof of Theorem **B**; this is simply because existential quantification does not distribute through conjunction.

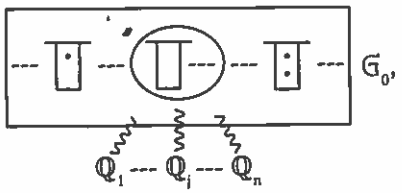
Some preliminary conventions are in order. First, in connection with our restriction on the quantifiers: we regard a quantifier $\mathbb{Q}$ as a *multivalent operator* that acquires specific valency only with respect to a **WFBG** and a target object label. Thus, we say that $\mathbb{Q}$ is *n-valent with respect to the pair* $(\mathcal{G}, x)$, $\mathcal{G}$ a **WFBG** and x an object label having exactly n open-hook occurrences in $\mathcal{G}$, if $\mathbb{Q}$ is either $\check{x}$ or $\hat{x}$.

Next, it will be convenient to have a special notation for the **WFBGs** we consider in what follows. If $\mathcal{G}$ is an open formula, we shall indicate it by a rectangle with a label, thus:

If M is a molecule occurring in $\mathcal{G}$, we shall denote it by a "roofed rectangle," thus: $\sqcap$. And if an open hook occurs in M we shall indicate it by a "bullet": • . Thus a partial representation of an open formula with a pair of open-hook-bearing molecules in it would appear thus:



If we now add quantifiers, it will be assumed (as is a fundamental guarantee in predicate logic) that $\mathcal{G}$ can be taken to be in *prenex normal form*, which for us means a sequence of mutually distinct quantifiers applied to an open **WFBG**. Thus, in our abbreviative notation, a quantified **WFBG** $\mathcal{G}$ would be skeletally represented by a diagram of the following type:



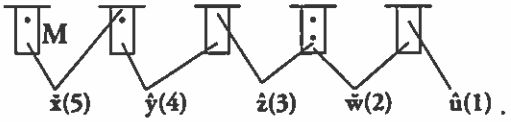
where $\mathcal{G}_0$ is open and $\mathbb{Q}_1$, ..., $\mathbb{Q}_n$ are quantifiers.

The following definition is key for our argument on behalf of Theorem **B'**:

Definition. Let $\mathcal{G}$ be a quantified **WFBG** in prenex normal form, and let M be a molecule of $\mathcal{G}$. By the $\mathbb{Q}$-*orbit of M in* $\mathcal{G}$ we mean the

set of all those molecules in $\mathfrak{G}$ that are linked to M by a sequence of quantifications, together with the linkages and "dead-end quantifications" (if any).

Here is a picture of a sample $\mathbb{Q}$-orbit, with M =, say, $\boxed{\cdot}$ in $\mathfrak{G}$:



(Notice that in this example, some of the molecules in the "open kernel" of $\mathfrak{G}$ would have valency > 2.)

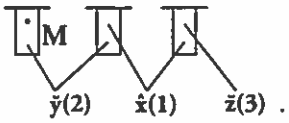
We shall agree that if M has no such linkages to other molecules, then it, together with any quantifiers acting on it, is its own $\mathbb{Q}$-orbit;

e.g., 

Observe that any two $\mathbb{Q}$-orbits M_{Q_1}, M_{Q_2} of $\mathfrak{G}$ are either disjoint or identical. One further terminological convention is useful:

Definition. By a *1-orbit* in $\mathfrak{G}$ we mean the $\mathbb{Q}$-orbit M_Q of a molecule M such that no molecule in M_Q other than M itself has an open hook, and M has *exactly one* open hook. (Thus, all other open hooks present in molecules of M_Q in the open kernel $\mathfrak{G}_0$ of $\mathfrak{G}$ have been closed by quantifiers of G.)

Example of a 1-orbit:



The following simple proposition will figure crucially in our impending proof:

Orbit Lemma. Let $\mathfrak{G}$ be a quantified **WFBG** in prenex normal form, such that $a(\mathfrak{G}) = \nu(\mathfrak{G}) = 3$, all molecules in the open part $\mathfrak{G}_0$ of $\mathfrak{G}$ are of valency ≤ 2, and all of the quantifiers acting on $\mathfrak{G}_0$ are of valency ≤ 2. Then $\mathfrak{G}$ contains at least one 1-orbit.

Proof. This is a straightforward consequence of the restrictions on the construction of $\mathfrak{G}$: if one of the valency-providing molecules, say M, of $\mathfrak{G}$ has valency 2 in $\mathfrak{G}$, then no linkages are possible for M, since otherwise it would have valency ≥ 3 in $\mathfrak{G}_0$; so M_Q = M, and the remaining valency-providing molecule M' in $\mathfrak{G}$ must be part of a 1-orbit. If, on the other hand, all valency-providing molecules of $\mathfrak{G}$ are of valency 1 in $\mathfrak{G}$, at most two of them can be linked in a single orbit, since none of them can have valency > 2 in $\mathfrak{G}_0$; and then the remaining one must be part of a 1-orbit. □

Theorem B'. Let $\mathfrak{G}$ be as in the preceding lemma, and let U be any universe of discourse with card(D) infinite. Then for any interpretation I_U, we have $I_U(\mathfrak{G}) \notin \{Ter, \tilde{T}er\}$.

Proof. To begin with, it's clear that we may concentrate on *Ter*: if $I_U(\mathfrak{G}) = \tilde{T}er$, then $I_U(\textcircled{\mathfrak{G}}) = Ter$; and $\textcircled{\mathfrak{G}}$ can be put in prenex normal form by the simple expedient of changing the outermost quantifier $\mathbb{Q}$ to its opposite, $\bar{\mathbb{Q}}$, and then slipping the negation oval between $\bar{\mathbb{Q}}$ and the rest of $\mathfrak{G}$, etc. until $\mathfrak{G}_0$ is reached. So, suppose it were the case that $I_U(\mathfrak{G}) = Ter$. Let $\{M_{1\mathbb{Q}}, ..., M_{k\mathbb{Q}}\}$ be the set of all $\mathbb{Q}$-orbits of $\mathfrak{G}$; by the orbit lemma, one of these, at least, is a 1-orbit. There are just the following two possibilities for $\mathfrak{G}$: a single 1-orbit along with an orbit containing two open hooks and also some number (possibly zero) of completely quantified orbits, or else three distinct 1-orbits plus, possibly, some fully quantified ones.

The advantage of orbital separation of open-hook labels is that it allows one to analyze the truth-value relationships of subformulas of $\mathfrak{G}$ without having to worry that a specific choice of D-values at the open-hooks in one orbital component will of itself force (via quantificational linkage) the choice of a D-value at such a hook in a different orbit. Influences between different orbits depend only on the *purely propositional structure of* $\mathfrak{G}_0$. We pause to illustrate this, and also to provide a hint why the rest of the argument goes as it does, by an example. Suppose $\mathfrak{G}$ were the following **WFBG** (strictly speaking, **WFBG** *form*):

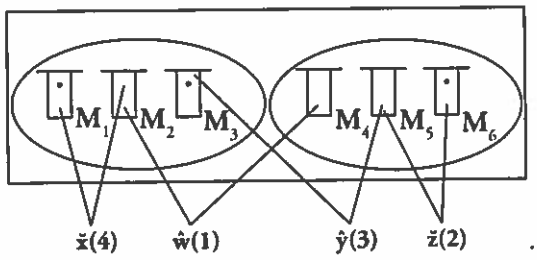
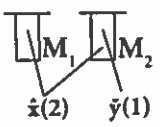


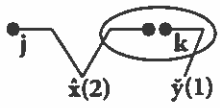
EXHIBIT A

For what we shall say about this, (•, •, •) can be whatever triple from D x D x D you please, subject only to the assumption that $\mathfrak{G}$ is true relative to I_U.

Note. There will, in general, be some "closed orbits" in $\mathfrak{G}$, with all available hooks quantified. We have not indicated any in Exhibit A, but for any such orbit there will be some finite number of possible *assignment types* for its molecular forms. Thus, for example, if



is such an orbit, one possible assignment type for its molecules, for a given I_U, would be $\exists x M_1(x)$ for M_1 and $\exists x \forall y \neg M_2(x,y)$ for M_2. Then if $\mathfrak{G}'$ =



, with  as M_1 and  as M_2, $\mathfrak{G}'$ may be either true or false relative to I_U, depending on whether x can have the same value for satisfying both conjuncts. In the discussion that follows, it is to be assumed that whatever assignment type has been chosen for such an orbit (of the finitely many such types available for that orbit), that type is such as to contribute to the satisfaction of (i.e., not prevent the satisfaction of) the overall formula $\mathfrak{G}(\bullet, \bullet, \bullet)$.

As has been hinted at in the above note, it will save some space in dealing with assignment types (though at the expense of pictorial clarity) to discuss Exhibit A in terms of linear, rather than betagraphic, notation. *Orbits*, to be sure, are clearer from the betagraphic presentation. Linearly, it says this:

$$\forall x\, \exists y \forall z\, \exists w[\neg(M_1(\bullet, x)\ \&\ M_2(x, w)\ \&\ M_3(\bullet, y))\ \&\ \neg(M_4(w)\ \&\ M_5(y, z)\ \&\ M_6(\bullet, z))].$$

We consider two possibilities for the 1-orbit, in this example.

1. It is the case that $\forall x \forall w[M_1(\bullet,x)\ \&\ M_2(x,w)\ \&\ M_4(w)]$. Then to ensure the correctness of $\mathfrak{G}(\bullet, \bullet, \bullet)$, the 2-orbit molecules can respond with any of the following:

1a. $\neg\exists y M_3(\bullet,y)\ \&\ \exists y \forall z M_5(y,z)\ \&\ \forall z \neg M_6(\bullet,z)$;
1b. $\neg\exists y M_3(\bullet,y)\ \&\ \forall z \forall y \neg M_5(y,z)\ \&\ \forall z M_6(\bullet,z)$;
1c. $\neg\exists y M_3(\bullet,y)\ \&\ \forall z \forall y\, \neg M_5(y,z)\ \&\ \exists z \neg M_6(\bullet,z)$.

2. It is the case that $\neg\exists w M_4(w)$. Here again, the 2-orbit has options to secure $\mathfrak{G}(\bullet, \bullet, \bullet)$:

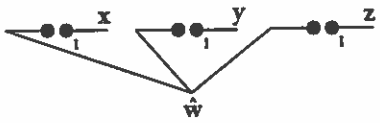
2a. $\neg\exists y M_3(\bullet,y)\ \&\ \forall y \forall z M_5(y,z)\ \&\ \forall z M_6(\bullet,z)$;
2b. $\neg\exists y M_3(\bullet,y)\ \&\ \forall y \forall z M_5(y,z)\ \&\ \neg\forall z M_6(\bullet,z)$;
etc.

Thus, even though the $\mathfrak{G}$ of exhibit A is a fairly simple **WFBG** form, there are a number of combinations of truth-value assignments that "work." Nonetheless, there are only *finitely* many, as is of course the case for *any* such $\mathfrak{G}$-form.[19] How many? We do not have to compute this number as a function of $\mathfrak{G}$ — it may be small, it may be large, depending on $\mathfrak{G}$ — we just have to know that it is finite, which it is. Let $E(\mathfrak{G})$ denote the set of all such assignments ("E" for "evaluation"), with $\text{card}(E(\mathfrak{G})) := e(\mathfrak{G})$. Each $a \in E(\mathfrak{G})$ can be split into a pair a_1, a_2

of subassignments, where α_1 is the part of a pertaining to the 1-orbit $M_{1,\mathbb{Q}}$ of $\mathfrak{G}$, and α_2 is the part covering the rest of $\mathfrak{G}$. (Again, refer to Exhibit A and the remarks following it, where we have given a *partial* listing of such subassignments.) Now back to the general $\mathfrak{G}$ posited for the theorem. Let *x, y, z* be the object labels attaching to the open-hooks in $\mathfrak{G}$, with *x* attached to the open hook of $M_{1,\mathbb{Q}}$; let an I_U be fixed; and let $\alpha_1, \alpha_2, \ldots, \alpha(\mathfrak{G})$ be all the assignments that can possibly correspond to triples $(\delta_1, \delta_1, \delta_1), (\delta_2, \delta_2, \delta_2), \ldots, \delta_{c(\mathfrak{G})}, \delta_{c(\mathfrak{G})}, \delta_{c(\mathfrak{G})})$ respectively, all $\mathfrak{G}(\bullet, \bullet, \bullet)$-correct assignments relative to I_U being accounted for. (These triples must indeed satisfy $\mathfrak{G}$ relative to I_U, since we are assuming $I_U(\mathfrak{G})$ = *Ter*.) Since D is infinite, there is a $d^* \in D$ different from all $\delta_1, \ldots, \delta_{c(\mathfrak{G})}$; and this d^* must also be such that (d^*, d^*, d^*) satisfies $\mathfrak{G}$ with respect to I_U. Moreover, it must do so under one of the assignments $\alpha_1, \ldots, \alpha_{c(\mathfrak{G})}$; say, under α_j. Taking α_j, then, as the truth-value assignment that accommodates $(\delta_j, \delta_j, \delta_j)$, split α_j into $\alpha_{j1}\alpha_{j2}$ as indicated above, and replace δ_j by d^* *in the 1-orbit* $M_{1,\mathbb{Q}}$ *of* $\mathfrak{G}$. Then nothing in the truth-value distribution changes from $\mathfrak{G}$ with $(\delta_j, \delta_j, \delta_j)$ substituted for (x, y, z) to $\mathfrak{G}$ with δ_j replaced by d^* *just in the* $M_{1,\mathbb{Q}}$ *orbit*: the interaction between α_{j1} and α_{j2} in rendering $\mathfrak{G}$ true is independent of exactly which of these two triples is occupying the open hooks of $\mathfrak{G}$. Since $d^* \neq \delta_j$, we have a contradiction from which the theorem follows, $\mathfrak{G}$ having been an arbitrary **WFBG** with the indicated properties. □

Concluding Remarks

(*1*) Once we allow quantifiers of $\mathfrak{G}$-valency ≥ 3, theorem B' collapses, as is only to be expected. (Note the remark in Herzberger 1981, that the **NRT** cannot survive contact with the full resources of the predicate calculus.) The following expression for *Ter*, using a predicate-logic form of Kempe's[20] "triple junction operator" (for us, just a quantifier of valency 3 relative to $\mathfrak{G}$) is doubtless the simplest example of this:



(*2*) The treatment of quantifiers as multi-valent operators taking specific valencies only in context does not seem widespread in the related literature, although it is implicit in Burch 1991, courtesy of his "HOOKID" operator. That is a little puzzling, since it seems to us a fairly obvious move. We suspect this may be connected with a tendency to view Peirce's "lines of identity" as being only about lines of bi-identity, thus failing to take into account ter-identical (or higher) junctions.

(*3*) Some words might be in order about our requirement that U be such that card(D)≥$\aleph_0$, everywhere except in Theorem **B**. First, it

is by no means evident that this requirement is actually necessary for Theorem **B'**[21]; but it is convenient for the proof. Second, we don't think Peirce would have objected to that assumption, since there are indications he would have insisted on infinitude for anything to be viewed, logically, as a "universe."

(4) Finally, for what sort of thing, beyond the arguments in this last section, might our approach prove useful? As hinted at in the Preface, we believe it should have descriptive value in areas where "small things get bigger via bonding, branching, and accumulation"; and that is quite a few areas. To repeat, our current focus of interest in this regard is on questions about neurons, in themselves and interactively, and how best to picture the relations involved in such things in as simple yet informative a way as possible. The molecular approach to relations would seem to be rather well-suited to such a purpose, as it might well also be, on a much larger scale, to the representation of various social interaction phenomena. One should also remember the foundational role of graphical relational logic in Semeiotic.

It will no doubt occur to some readers to ask: since we are altering the treatment of quantification from Peirce's "lines of identity" to our "projective" notation, what advantages do we feel this change has? They are, we propose, at least two-fold. First, there are no longer any problems of sorting out the nuances of cut-crossings. Second, our treatment facilitates the simplification of proofs for such things as the status of "non-reduction" vis-à-vis quantification (see Theorem **B'**).

Our suggested Betagraphic approach to bonding and relations as a foundation for studies of semeiosis may provide a new and easier technique for representing the various interactive neural and cognitive systems currently being explored in neuroscience, psychology, and particularly the emerging field of neuroesthetics. The application of our representational framework as a form of diagrammatic modeling[22] promises to offer some additional insights into Peirce's ideas of diagrammatic thought as a genuine depiction of "moving pictures of thought," "moving perception," or "reasoning in action."

Further, in regard to the foundational role of graphical relational logic in Semeiotic (Peirce's *Science* of semeiosis), we note: the importance of this role can be appreciated if we recall his basic notion that every *sign* (in the sense of an instance of semeiosis) necessarily involves a relation of the triadic type. Thus his study of semeioses presupposes the logic of relations. The essential role of triadic relation types in semeiotic is well illustrated by remarks such as those found in the *Collected Papers* (for example, see 1.480, 1.540, 2.242, 2.274, 4.447); in those locations, and in similar discussions, Peirce indicated that a sign (in the sense of representamen) is a representation of an object to an interpretant. Here Peirce was describing a situation in which there are three items—object *o*, representamen *r*, interpretant *i*—participating as relata in a relation **T** of the genuinely triadic type.

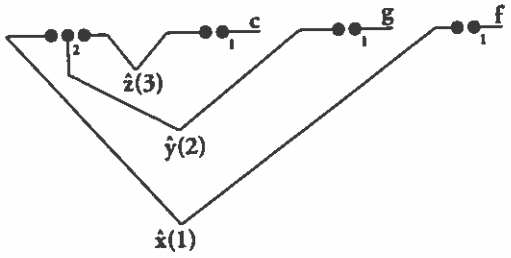
Such a relation as **T** can be depicted in Betagraphic or Beta EG; or it can be described in linear notation, where it would appear, in a propositional function form, as **T**(*o,r,i*) in which **T** is a genuine triadic relation involving three relata: *Object o*, *Representamen r*, and *Interpretant i*. **T** is understood as the relation that describes the fact that *r* represents the object of discourse *o* to the interpreting function *i*. At this level of abstraction, **T** is capable of assuming further subtypes of triadic semeioses, such as Icon, Index, Symbol, Dicisign, and so on throughout the full list of subtypes explored by Peirce. Moreover, *o*, *r*, and *i* may acquire more definite specification in terms of concrete cases: for example, "Smoke at grid location 2 represented the presence of fire at grid 2 to Ranger Ike watching in the tower" (here **T** is further specified as the semeiosis sub-type *Index*, wherein for this *concrete* case it is further specified that "fire at grid 2" is *o*, "smoke at grid 2" is *r*, and *i* is Ike's knowledge that "smoke is caused by fire"). A nonconcrete or general description of an indexical sign would be expressed as "There is a triadic relation **T**(*o,r,i*) wherein *r* is interpreted as the effect of a cause *o*, *i* being the interpreting function '*o* causes *r*'." We could add existential (or other) quantifiers for some or all variables in this expression, in either a more generalized form, or as in the concrete case above, thus (*f* fire at grid 2, *g* smoke at grid 2, *c* Smoke is caused by fire):

$$\exists x \exists y \exists z\, [x=f \,\&\, y=g \,\&\, z=c \,\&\, \mathbf{T}(x,y,z)].$$

A similar treatment for other types of semeiosis may be constructed using either Betagraphic or Beta EG or linear predicate logic notation.

What would the "Ranger Ike" example look like in Betagraphic? In its most general form, with relata *o*, *r*, *i*, we would simply write, say,

In the more specific case of smoke, fire, and Ike, with the indicated quantifications, it would appear as, say,



All the rules and theorems available in any of the three mentioned systems for logic of relations, with bonding made explicit, will guide the principles of semeiotic. A classic example of one such principle of semeiotic is "Every sign [semeiosis] must be interpreted in another

sign [semeiosis]." This is a case wherein Peirce applied **NRT** within the Logic of Relations (no matter whether expressed as Betagraphic, Beta EG, or linear predicate calculus) to the flow of sign action, or *Semeiosis*. The supporting argument for this classic example of a result obtained in semeiotic, by using the logic of relations, may be outlined in the following manner:

> (*prior result*): **NRT** is correct, which we take as a working hypothesis of semeiotic.
>
> (*i*): Any further interpretation of a prior semeiosis (which will incorporate a triadic relation type) will be executed in a communicational event (another semeiosis).
>
> (*ii*): All communicational events are semeioses embodied in triadic relation types.
>
> (*iii*): Thus, any further interpretation of a prior semeiosis (a triad) must be embodied in another semeiosis also employing a triadic relation type (otherwise **NRT** would be violated).

The expression of semeiotic in predicate calculus may look a bit foreign at first sight. That feeling is perhaps due to the role of bonding not having been noticed or overtly used in those circumstances. In Betagraphic (or Beta EG) bonding operations are more prominently employed and acknowledged. It is a fact that bonding is equally, if implicitly, present and important in algebraic forms for expressing the logic of relations. Bonding operations make especially clear that quantification, no matter where employed, is a valency-involved procedure. This recognition may clarify the role of the logic of relations in the foundations of Semeiotic in a way that is useful for scholars who prefer working in the linear mode of the predicate calculus.

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NOTES

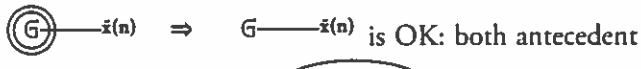
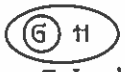
1. Thomas McLaughlin (lead author), Elize Bisanz, Scott R. Cunningham, Clyde Hendrick, Levi Johnson, Kenneth Laine Ketner, and Michael O'Boyle.

2. On Peirce's use of "weld," see Peirce 1992 (1898), 42, 91–92, 95, 158, 159, 160.

3. A child's creative construction set involving wooden or plastic hubs and links; it bears similarity to sets used to construct models of molecules in chemistry courses.

4. Burch 2011 is very helpful about this.

5. Peirce, ever the scientist and fallibilist, encouraged future research and improvements for his basic approaches in Existential Graphs, especially as applied to Semeiotic: see *CP* 4.510–529 (made public in the Lowell Lectures of 1903) and *CP* 4.573–584 (made public before the National Academy of Sciences, April 1906 meeting in Washington, D.C.).

6. In connection with double-oval *removal*, we mention the following important points: (*1*) it is allowable, for the removal of  , for the double oval to be *crossed by* a line (or lines) of quantification, but *not* generally allowable for it to be *split* by one. Thus, for example,  is OK: both antecedent and consequent say "for every *x*, we have G." But  is *not* OK; *it* says "if there is an *x* such that G, then for every *x*, G," a claim that is more often than not false. If we replaced (*n*) by (*n*) in this last *bad* implication, we would get a correct implication saying "if for every *x* we have G, then for *some* *x* we have G." As a practical matter, it is safest to simply avoid removing a double oval *split* by a line (or lines) of quantification. (*2*) Under no circumstances may  be replaced by either G 𝔥 *or* G  , 𝔥 being a **WFBG**.

7. In this instance, "occurrences" refers not only to open-hook occurrences, but also to occurrences on bonding tags and/or in quantification. Thus, in particular, we may perform a change of variable *in a sentence*, an operation that is sometimes useful.

8. Recall from note 6 that we are not permitted to remove a double negation if there is a **WFBG** *intervening between* the two negation ovals. Again, the only intervening mark(s) definitely not interfering with removal of  would be one or more lines of quantification *crossing both ovals*.

9. See, e.g., Mendelson 1964.

10. In the event that *x* does not have any open-hook occurrence in 𝔥 either, (*n*) is a "vacuous quantifier," and (since it then has nothing to attach to in 𝔥) simply does not appear in the graph. But for some general commentary on vacuous quantifiers, see paragraph *2* of the Postscript.

11. See Burch 2011 for an intensive treatment of the Peircean line of identity.

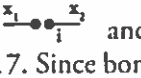
12. See Interdisciplinary Seminar on Peirce 2011.

13. The assignments to atoms in clauses (2) through (4) depend only on the indices (subscripts) of the atoms, not on the particular object labels that may be represented by *x*, *y*, or *z*. Thus, e.g., $I_U(\bullet^{x_i}_{i}) = I_U(\bullet^{x_k}_{i})$ for all *i*, *j*, *k*. Also, as is customary, we denote the empty set by "ø."

14. Both here and in (*2*) above and (*4*) below, we have cases in which the atom in question is a nullecule, and we have said in Part I that these are to have the status of sentences. So, how should we understand, say, $\bullet^{c}_{i}$ or $\overset{c}{\bullet}\!\!-\!\!\bullet^{d}_{s}$, when read as a sentence, apart from its function as a name in the present definition? (This "name function" relative to I_U cannot interfere with what happens under bonding, since nullecules cannot bond. $\bullet^{c}_{i}$ should be read as "*c* is in the universe," $\overset{c}{\bullet}\!\!-\!\!\bullet^{d}_{s}$ should be read as "(*c*, *d*) is in the relation $\overset{x}{\bullet}\!\!-\!\!\bullet^{y}_{s}$," and similarly for all

other "constant nullecules." (In particular, we would read  , as a sentence, as "$c = d = e$," with a naming value of ø in I_U if this is false in U under I_U.) Most of these sentential readings, of course, have their truth-values dependent on U and I_U.

15. Notice that here, when we speak of an "n-ary relation" and denote it by $R(\mathbf{x}_1, ..., \mathbf{x}_n)$, we are referring by n to the valency of R, were it presented graphically, as we shall shortly be doing.

16. Bonding may result in a molecule,  , say, that under some interpretation I_U yields via ˜-evaluation in D a pair such as (10, 7) as part of the relation it names; in such a case, one x is delivering a value of 10, the other a value of 7. This may, understandably, be disconcerting to some readers. To find out how this happened, one need only examine the reconstruction tree. If one is too uncomfortable with its happening, one can make sure that the open occurrences of "x" in one of the items to be bonded has been replaced by some other object label prior to bonding, this without disturbing the I_U-assignments. (In our example, this question only arises when $y \neq x$.) Indeed, we can make sure the whole reconstruction tree is relabeled to prevent this kind of awkwardness in the semantic reading versus the syntax. (Note: I_U assigns the same relation to both of, say,  and  ; see note 13.

17. Since bonding deletes the common object label of the bonded hooks; here the result of the bonding has no occurrence of that or any other label on an open hook.

18. By a sentence being true, or true in D, we here, of course mean that it is true in the universe $U = < D, 2^D, ... >$ under an interpretation I_U. But here only card(D) ≥ 2 matters.

19. Relative to a given form $\mathfrak{G}(\bullet, \bullet, \bullet)$, quantificational linkages may exert influences between different molecules *in the same orbit*. E. g., $\exists y M(\bullet, y)$ may be true for a given I_U value of •, but if the existential quantifier links M to $\forall z M'(y, z)$, it may be that no y that works in M also works in $\forall z M'(y, z)$ for that value of •, though under some other choice for • we could have $\exists y(M(\bullet, y) \,\&\, \forall z M'(y, z)$. Nevertheless, the total number of possibilities for $\mathfrak{G}(\bullet, \bullet, \bullet)$-satisfaction (i.e., assignments with such value-choices and linkage influences taken into account) is finite.

20. For an excellent account of A. B. Kempe's work in its relation to Peirce, A. Cayley, J. J. Sylvester, W. K. Clifford, plus J. Royce and his students N. Weiner, C. I. Lewis, H. M. Sheffer, and R. Eaton, see Grattan-Guiness 2002; see also Kempe 1886.

21. It is worth noting, in any case, that what is actually critical for theorem **B'** is the conclusion of the Orbit Lemma: both the 2-valent restriction on quantifiers and the restriction on pre-quantificational molecular valencies can be dropped, *if we assume that our **WFBG** has a 1-orbit.*

22. Recent works in image studies attracted to Peirce's notion of diagrammatic thought have led to a grave misunderstanding of Peirce by focusing on his doodles and pen tests as genuine examples of his theory of diagrammatic thought; see Engel et al. 2012. For Peirce's account of diagrammatic thought, see Ketner 1984.