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How Hintikka Misunderstood Peirce's Account of Theorematic Reasoning

Jaakko Hintikka, in "C. S. Peirce's 'First Real Discovery' and Its Contemporary Relevance," an article from one of two recent issues of *The Monist* on "The Relevance of Peirce" (1980), has proposed that the kind of deductive reasoning that Peirce called theorematic can be understood in the following way: "a valid deductive step is theorematic if it increases the number of layers of quantifiers in the propositions in question" (p. 307). This statement, however, is incorrect as are related descriptions of Peirce by Hintikka. I agree with Hintikka that the importance Peirce associated with his first real discovery about the nature of mathematical method is fully justified (p. 312). But I shall argue that Hintikka has failed to describe Peirce's account of the matter, and because of that, he has not correctly understood why the discovery is important.

First consider that the title of Hintikka's article is misleading, suggesting as it does that it is about a "first real discovery," period. The actual context makes clear that it is Peirce's first real discovery about mathematical method, a subject upon which he worked consistently throughout his life. Although Peirce constantly referred to this, until recently perhaps only Carolyn Eisele (see 1979, a collection of her essays, ca. 1950-present) had realized the importance of this fact for gaining a correct understanding of all of Peirce's work (Ketner 1982). Theorematic reasoning is one of two kinds of deduction, a distinction which Peirce claimed to have discovered. Here is a clear statement of the matter from a letter to William James, written in 1909 (Eisele 1976, 3:869):

> There are two kinds of deduction; and it is truly significant that it should have been left for me to discover this. I first found, and subsequently *proved* [this is probably a reference to "On the Algebra of Logic: A Contribution to the Philosophy of Notation," *American Journal of Mathematics,* 7: 1885, 180-202, and to its predecesser, Peirce 1868b], that

> every Deduction involves the observation of a Diagram (whether Optical, Tactical, or Acoustic) and having drawn the diagram (for I myself always work with Optical Diagrams) one finds the conclusion to be represented by it. Of course, a diagram is required to comprehend any assertion. My two genera of Deductions are 1st those in which any Diagram of a state of things in which the premisses are true represents the conclusion to be true and such reasoning I call *corollarial* because all the corrollaries that different editors have added to Euclid's Elements are of this nature. 2nd kind. To the diagram of the truth of the Premisses something else has to be added, which is usually a mere May-be and then the conclusion appears. I call this *theorematic* reasoning because all the most important theorems are of this nature.

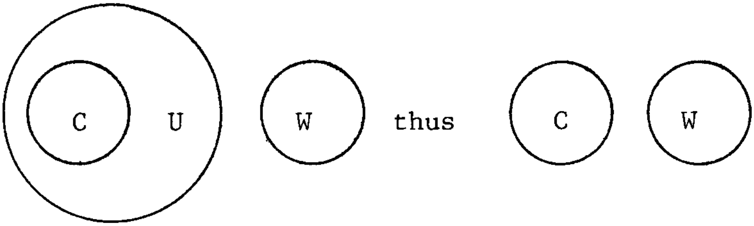
Elsewhere (*Collected Papers,* 3.418-419) we find that auditory diagrams separate their parts in time, whereas visual diagrams separate them in space, and that algebra is a kind of diagram, and that speech is such an algebra, from which we can infer that Peirce regarded speech as a kind of diagram. Another useful statement shedding light on this issue was part of the 1903 lectures on pragmatism (Eisele 1976, 4:164): "I am satisfied that all necessary reasoning is of the nature of mathematical reasoning. It is always diagrammatic in a broad sense although the wordy and loose deductions of the philosophers may make use rather of auditory diagrams, if I may be allowed the expression, than [of] visual ones." In many other places, early and late, Peirce stated that all mathematical reasoning is diagrammatic. Here are two good instances:

> *The Reasoning of Mathematics* . . . The first things I found out [again probably a reference to the 1885 article in the *AJM,* and related pieces] were that all mathematical reasoning is diagrammatic and that all necessary reasoning is mathematical reasoning, no matter how simple it may be. By diagrammatic reasoning, I mean reasoning which constructs a diagram according to a precept expressed in general terms,

> performs experiments upon this diagram, notes their results, and expresses them in general terms. This was a discovery of no little importance, showing as it does, that all knowledge comes from observation. (Eisele 1976, 4:47-48)
>
> The procedure of the mathematician is, first, to state his hypothesis in general terms; second to construct a diagram, whether an array of letters and symbols with which conventional "rules," or permissions to transform, are associated, or a geometrical figure, which not only secures him against any confusion of *all* and *some*, but puts before him an icon by the observation of which he detects relations between the parts of the diagram other than those which were used in its construction. This observation is the third step. The fourth step is to assure himself that the relation observed would be found in every iconic representation of the hypothesis. The fifth, and final, step is to state the matter in general terms. (MS 1147, published at Eisele 1976, 3:749)

Here, then, is the true importance of Peirce's corollarial/theorematic reasoning distinction – it makes significant contributions toward showing that mathematics and logic are observational, experimental, hypothesis-confirming sciences, in which one makes hypotheses about, and observes and experiments upon, diagrams according to a distinctive method. To aid in understanding how these distinctions function in practice, I shall provide some examples of the method described in the above quotations, first of corollarial reasoning, then of theorematic reasoning.

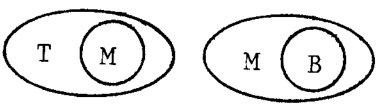
Let us begin with this argument: "No unusual persons are welcome at Mrs. Dixwell's school; All crocodile wrestlers are unusual persons; therefore, No crocodile wrestlers are welcome at Mrs. Dixwell's school." Following the five steps of mathematical method Peirce outlined, we hypothesize that if the premisses of this argument were true, the conclusion would also be true. Using the method of Euler, we construct a diagram of the premisses on the hypothesis that they are true, thus (adding the conclusion's diagram also):



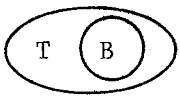
This diagram is an icon of the argument, in that some of the spatial relations within it are identical in structure to the relations in the argument under study, given that we understand Euler's terminology. With such a diagram, we have gained a significant advantage, in that we can observe it with our most competent sense, sight. In such an observation we need only inspect the diagram of the premisses to see that the diagram of the conclusion is directly presented therein. The fourth step is to be sure that every such representation of the hypothsis would produce identical results, something which in this case is guaranteed by the techniques and conventions of forming the Euler diagram. The fifth step is to generalize the results, which we can easily do by reflecting that these relations would hold no matter what the particular classes were. Thus, we have, through corollarial reasoning, discovered a general relationship which logicians have called Celarent.

We turn now to an example of theorematic reasoning. In studying this argument, let us use the alpha part of Peirce's Existential Graphs (for an account of this part of EG see Ketner 1981a): "If one is a truffle hunter, one will have muddy boots; If one has muddy boots, one cannot enter the Blue Ox Pub; therefore, If one is a truffle hunter, one cannot enter the Blue Ox Pub." Again the hypothesis is whether the premisses being true require the conclusion to be true. Here is an EG diagram of the premisses:

1.

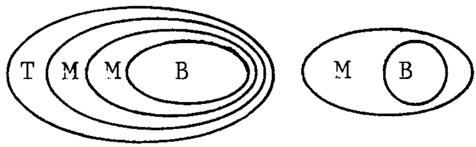


And here is an EG diagram of the conclusion:



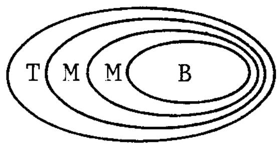
Notice that we cannot detect the conclusion diagram through a mere direct observation of the premiss diagram. This will then be an example of theorematic reasoning, in which an experiment upon the premiss diagram will be needed. So, not knowing in advance that the next few experimental steps will produce anything except a blind alley or a failed strategy, we might begin by transforming the premiss diagram in the following ways, using the five transformation rules of alpha EG:

2.



1, IT

3.

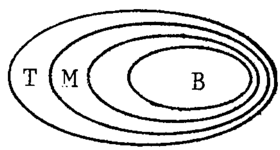


2, E

Line 3 is a mere "maybe," an experiment, which may or may not be fruitful. It is important to note here that production of experiments within theorematic reasoning, on Peirce's view, is done through abduction, the kind of reasoning that results in generation of a hypothesis for future test. This marks yet another similarity between the methods of logic or mathematics and of science in general (physics, for instance). One more similarity between the methodology of mathematics and physics is the use of the pragmaticistic maxim as a central tool in experiment design. In the case of the argument examples presented here, the pragmaticistic maxim helps one to recognize when an experiment is successful or unsuccessful. Basically, if a hypothesis manipulated in a manner consistent with the principles and terminology of the system being used produces progress toward a solution, it is confirmed; if no progress or a violation of the system occurs, it is not confirmed. A difference between mathematics/logic and sciences like physics is that the content of the former is totally hypothetical, whereas in physics one wants to know if one's hypothesis is true of nature, not simply consistent with a system of hypotheses, definitions, and rules.

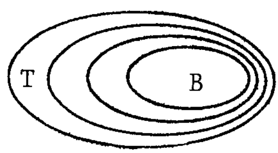
Having achieved the last graph above, a trained user of EG (just like a trained user of Euler's diagrams) would be able to perceive that the conclusion is directly presented in it, which is shown by the following corollarial sequence which completes the confirmation of the original hypothesis.

4.



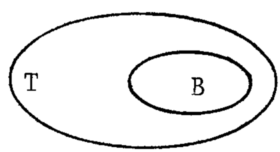
3, DE

5.



4, E

6.



5, DC

Thus, the experiment introduced in line 3 is a success.

This last is also a general result due to the nature of the techniques and terminology of EG, and we see that the argument's general form is true of any sentences connected by means of these logical constants. Hence, by observing tokens, we have achieved a knowledge of a general type, in this case that which logicians call "Hypothetical Syllogism." We should note for future reference that the above procedure is a kind of induction. Notice that there are no quantifiers in any proposition in this argument.

We shall consider one last instance of theorematic reasoning, this time using an algebraic diagram ("an array of letters and symbols," which is not a "geometrical figure"). Here is a sample argument:

> "If Zina is present, then Juliette is absent; Either Zina is present, or Juliette is absent; therefore, Juliet is absent."
>
> Following Peirce's method, we create a diagram of the hypothesis: (Z implies J) and (Z or J), therefore J

An inspection of the premisses does not enable one directly to observe the conclusion therein. So, we shall use theorematic reasoning and design an experiment. First, the experimental hypothesis. We suspect that if we change the second premiss to an equivalent implicational form, then we will have a hypothetical syllogism which will produce "J or J" eventually, which if so would yield the conclusion by another equivalence. Thus, we transform the second premiss to "–Z implies J." That, however, fails to give a hypothetical syllogism. So, we try a second experiment by leaving the prior transformation in place and contraposing the first premiss, to yield "–J implies –Z." That does enable a hypothetical syllogism when coupled with the changed first premiss, to give "–J implies J." Thus the experiment has succeeded, and we now have a corollarial move, because we clearly know that "–J implies J" is equivalent to "J or J," which is equivalent to "J." Again, there are no quantifiers in these propositions.

Now it should be clear that for Peirce, any kind of algebraic formula is a diagram just as a figure, say, from Euclid, would also be. Thus Peirce is not a constructionist of the kind Hintikka describes (p. 306). And he did not reach his account of theorematic reasoning by generalizing from Euclid, as Hintikka asserted (pp. 305-6). (Euclid's work simply provided Peirce with an elementary and widely understandable *example* of the figurative – as opposed to the algebraic – kind of diagrammatical reasoning, an example which he used for didactic purposes.) Contrary to Hintikka's suggestion, Peirce reached his account of theorematic reasoning from a long study of the nature of mathematics and of mathematical method, which included a thoroughgoing analysis of the basis of signs in mathematics (consider Peirce 1868a,b,c,d,e, 1885, for some of the principal studies). The concept of construction that Hintikka wants to impose upon Peirce is that associated with figures like those of Euclid, but this is not appropriate because Peirce talked of constructing (better, creating) diagrams in a very general sense, which includes figures as well as such other things as algebras, maps, pages in books, or language. Moreover, to anticipate another possible misunderstanding, Peirce's "diagrammatic thought" is not the same as the constructionism associated with the school of Brouwer, although it has some similarities, yet lacks its shortcomings (Levy 1982). One shortcoming of the Intuitionist school that is not to be found in Peirce's philosophy of mathematics is the Intuitionist rejection of reasoning involving infinity. Peirce was among the first to show how we can reason effectively about infinity (for an early example from 1859, see Fisch, ed. 1982: 37-43).

Another way to appreciate Peirce's points is to consider the kind of view that Hintikka proposes (p. 306) as a hypothetical objection to Peirce's supposed Euclidean constructionism. Hintikka's suggested objection to Peirce's supposed constructionism is that such constructed figures are dispensable in geometrical and other valid deductive arguments, since geometry can be formalized and expressed symbolically or algebraically. Peirce's answer to this, I am sure, would have been: "But the algebras or symbolical expressions with which you replace Euclid's figures are themselves diagrams, and the conclusions that mathematicians or logicians draw from them are done through visually observing these algebras or formal marks and the relations they contain, and through experimenting upon them, and through having reached confirmed hypotheses about them – if all visual (or any other observational/perceptual) elements were removed, there would be no basis upon which to engage in mathematical or logical activity." Hintikka, on the other hand, proposes to answer the suggested objection to Peirce's supposed constructionism by saying (p. 307): "The new individuals do not have to be visualized, as the geometrical objects introduced by an Euclidean construction are. They have to be mentioned and considered in the argument, however." To that a Peircean might respond, "Close off all your observational capabilities, and you will not be aware of either a mention or an argument, for mentions are sounds or marks, as are arguments, and it is these arrayed sounds or marks and the relations they exhibit – diagrams in short – that are the realities we observe (perceive) and reason upon in logic or mathematics." (By the way, Peirce was fully aware of developments such as topology, the work of Riemann, and the Erlanger Programm – see Eisele 1979.)

This brings us to the central claim of Hintikka's essay – that theorematic reasoning "is an increase in the number of layers of quantifiers in the propositions in question." Hintikka invokes this option as a way of responding to the hypothetical anti-constructionist view he introduced, thinking that by doing so he has saved the day for Peirce. But the proposed face-saving move is not one that Peirce needed, nor one that is acceptable to his approach. Indeed, if Peirce had wanted to make the distinction Hintikka suggests, he could have easily done so, for Peirce (with O. H. Mitchell), as is well known, was an independent inventer of quanitification in logical algebras (see Brunning 1981; Mitchell 1883). Moreover, if quantification is essential to the distinction (as Hintikka claims), then theorematic reasoning should not be possible in calculi or systems that do not include the notion of quantification. The example of the Blue Ox Pub and Juliette's absence are cases in point. Peirce's distinction is applicable to any mathematical or logical system, independently of whether the system contains quantification. Quantification has nothing to do with Peirce's distinction; on the other hand, experiment and observation are at the heart of it.

Hintikka's claim (p. 307) that his quantificational-layering proposal is "the closest rational reconstruction of those explanations of Peirce's which are couched in terms of the need 'to experiment in imagination upon the image of the premiss' in a theorematic deduction" can now be seen to be simply mistaken. It is not the closest rational reconstruction. It is a false interpretative conclusion based upon a failure to understand Peirce's actual points, points which Peirce plainly stated. And the significance of Peirce's points in this matter lie in the direction of working out a successful philosophy of mathematics, especially trying to resolve the question of the nature of mathematical method. Peirce's answer was that mathematics is an empirical science. That this is very defensible in the form that Peirce presented it is a thesis I cannot now explore (however, see Levy 1982), but that is the direction we should look to see why Peirce was so excited about theorematic reasoning. A short-hand description of the source of his excitement could be: "It shows that all knowledge is unified in terms of the general method of science, which is observational, hypothesis-confirming, and objective, for even mathematics, seemingly a non-experimental science, upon closer study, exhibits those same features in its method."

One who wishes further to pursue this path that Peirce developed will find that some other concepts, perhaps as despised as that of diagram, play significant roles as well, chief among them being "abstraction" (Ketner 1983; Hawkins 1981). Even the fashionably suspect notion of "mental image," recently provided with a new empirical basis (Block 1981), will be seen to have a viable place in Peirce's approach to diagrammatic thought, especially in its new empirical dress.

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